POVZETEK

Namen diplomskega dela je predstavitev osnov spektralne teorije grafov in prikaz uporabe orodij linearne algebre pri spektralni analizi grafov. Hkrati se z obravnavo različnih družin grafov in primeri uporabe v drugih vejah znanosti ustvari zaokrožena slika o tej relativno mladi veji teorije grafov. Diplomsko delo je razdeljeno na štiri poglavitne dele. V prvem delu so predstavljene osnove linearne algebre s poudarkom na algebri simetričnih matrik in osnove teorije grafov, ki so pomembne za nadaljnje razumevanje. V drugem delu so pokrite bistvene ugotovitve spektra grafov, od osnovnih definicij grafovskih matrik in lastnosti spektrov najbolj znanih izmed njih, matrike sosednosti in Laplaceove matrike, do najbolj pomembnih ugotovitev, povezanih s spektri matrik, ki smo jih razdelili v smiselne vsebinske sklope. Tako so predstavljeni spektri komplementarnih grafov, povezava med matriko sosednosti in sprehodi, pomen Laplaceove matrike pri štetju vpetih dreves in povezava med Laplaceovim spektrom in številom povezanih komponent grafa. Tretji del je posvečen nekaterim družinam grafov, katerih spekter ni trivialen in zahteva uporabo orodij tudi iz drugih področij analize in algebre. Med drugimi so to kartezični produkt grafov, grafi povezav in krepko regularni grafi, pri čemer pa so zaradi celovitosti obravnavani tudi tisti najbolj znani, kot so na primer regularni grafi. V zadnjem delu je poudarek na širši uporabi spektralne analize, ki ima mnogo aplikacij. V tem delu sta podrobno obravnavani dve: Wienerjev indeks s področja kemije, kot prvi vpeljan in najbolj uporabljan topološki indeks in pa mere središčnosti ter algoritem PageRank za rangiranje spletnih strani kot najpogosteje uporabljana aplikacija.

Ključne besede: lastna vrednost, determinanta, teorija grafov, spekter grafa, matrika sosednosti, Laplaceova matrika, Wienerjev indeks, omrežje, mera središčnosti, PageRank

ABSTRACT

The purpose of this thesis is to present the basics of spectral graph theory and to demonstrate how the tools of linear algebra are used in spectral analysis of graphs. At the same time, a complete picture of this relatively young branch of graph theory is created by means of analyzing various families of graphs and examples of use in other branches of science. The thesis is divided into four main parts. The first section provides basic information about linear algebra with the emphasis on algebra of symmetric matrices, as well as basic overview of the graph theory, which serve as background knowledge for understanding the main part of the thesis. The second part of the thesis presents the main findings related to the graph spectrum, beginning with the basic definitions of graph matrices and spectral properties of the adjacency matrix and Laplacian matrix, concluding with the most important characteristics related to the matrix spectrum, which are divided into logical units. These are the spectrum of graphs' complements, walks and their relationship with the adjacency matrix, the role of Laplacian in counting spanning trees and the connection between the spectrum of Laplacian and the number of connected components in a graph. The third part is about families of graphs whose spectrum is not trivial and requires the use of tools from other areas, such as analysis and algebra. These families are, among others, the Cartesian product, line graphs and strongly regular graphs. For the purpose of integrity, some other well described graphs, such as regular graphs, are examined in the thesis. The last part of the thesis focuses on the wider use of spectral analysis which has many applications. In this section two of them are discussed in detail: the Wiener index in the field of chemistry, as the first and most widely used topological index and centrality measures, specifically the PageRank algorithm for ranking websites, as one of the most frequently used applications.

Keywords: eigenvalue, determinant, graph theory, spectra of graphs, adjacency matrix, Laplacian matrix, Wiener index, network, centrality measure, PageRank

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