Geometry of Integrable Systems

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Abstract

The topic of this thesis are finite dimensional Hamiltonian integrable systems and certain aspects of symplectic geometry of their underlying phase spaces. The main result is presented in Chapter 3. The complete integrability of a class of Hamiltonian systems (T^*M, ω_{can}, H) is proved, where M is an arbitrary compact or non-compact Riemannian symmetric space. This class contains some classical examples of integrable systems such as C. Neumann's system and the spherical pendulum. The new examples we consider are motion on projective spaces, which in turn yield integrable motions on spheres subject to certain quartic potentials. Symplectic reduction gives the integrability of the motions of a particle on $\mathbb{CP}^n$ and $\mathbb{HP}^n$ in a quadratic potential field with the additional presence of magnetic and the Yang-Mills fields respectively. The connection of the systems (T^*M, ω_{can}, H) with Nahm's equations is investigated. It is also indicated how these systems fit into the context of Hitchin's integrable systems on $T^*\mathcal{M}_{par}$, where $\mathcal{M}_{par}$ is the moduli space of stable parabolic structures on $G^{\mathbb{C}}$-principal bundle over a complex curve C.

Hitchin's systems on $T^*\mathcal{M}_{par}$ are studied in Chapter 2. In a different way, these systems were already studied by E. Markman. Our approach allows us to obtain a family of symplectic structures ω_{λ_D} on $T^*\mathcal{M}_{par}$ parametrised by a set $\mathcal{S} \subset \bigoplus_{i=1}^r \mathfrak{h}_i$ where D is the divisor of marked points with $degD = r$ and $(\mathfrak{h}_i)^* \subset (\mathfrak{g}^{\mathbb{C}})^* = (Lie(G^{\mathbb{C}}))^*$ are duals of Cartan subalgebras. The set $\mathcal{S}$ consists of point $\lambda_D = (\lambda_1, \dots, \lambda_r)$ such that λ_i are regular and of the point $0 \in \bigoplus_{i=1}^r \mathfrak{h}_i$. For every symplectic space $(T^*\mathcal{M}_{par}, \omega_{\lambda_D})$ we construct an integrable system. The system corresponding to $\lambda_D = 0$ (the case studied by Markman) is exceptional in our family. We show how this is expressed in terms of spectral curves.

The main topic of Chapter 1 are real symplectic structures on complex coadjoint orbits $\mathcal{O}^{\mathbb{C}}$. The orbit $\mathcal{O}^{\mathbb{C}}$ has real Kostant-Kirillov forms and the canonical cotangent form since $\mathcal{O}^{\mathbb{C}} \cong T^*\mathcal{O}$ for some compact orbit $\mathcal{O}$. The two are compared via the mechanical connection construction. This comparison is generalised to the case where $\mathcal{O}^{\mathbb{C}}$ is replaced by an $\mathcal{O}^{\mathbb{C}}$-fibre bundle.

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Contents

Chapter 0

Introduction

0.1

The term integrable systems denotes systems of differential equations, usually describing some physical situation, which can in some sense be solved. In this text we will be concerned with the integrable systems corresponding to ordinary differential equations or in other words, to dynamical systems which are integrable. Since there is some ambiguity in what is meant by integrating a system of differential equations, many definitions of integrability are possible. This is particularly so in the case of the systems of PDE's. In the case of ODE's however the situation is simpler and there is more consent on what integrability is. The systems that we will consider are Hamiltonian. This means that the space of states $\mathcal{X} \times \mathbb{R}$ is a manifold of dimension $2n+1$ with a local chart $(q_1, \ldots, q_n, p_1 \ldots p_n, t)$ around each point. The space $\mathcal{X}$ is called the phase space. There is a function $H : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ called the Hamiltonian, and the relevant ODE's in the local coordinates are

$$\begin{aligned} \dot{q}_i &= \tfrac{\partial H}{\partial p_i} \\ -\dot{p}_i &= \tfrac{\partial H}{\partial q_i} \end{aligned} \tag{1}$$

for $i = 1, \ldots n$. The variable $t \in \mathbb{R}$ represents the time, while q_i's and p_i's are often, but not always the coordinates of position and of momentum. Typically H is of the form $H(p,q) = \|p\|^2 + V(q,p)$, where $\|p\|^2$ is kinetic and $V(q,p)$ potential energy. A frequent stipulation $\frac{\partial H}{\partial t} \equiv 0$ then means that the total energy H of the system is constant with respect to time, i.e. the system is conservative. The coordinates (q,p) are called the canonical coordinates, and the system of equations 1 the canonical equations. Every system of coordinates on $\mathcal{X}$ in which the system 1 preserves its form is called canonical. The vector field $\xi_F(x) = \sum_{i=1}^n \frac{\partial F}{\partial p_i} \cdot \frac{\partial}{\partial q_i} + \frac{\partial F}{\partial q_i} \cdot \frac{\partial}{\partial p_i}$ is called the Hamiltonian vector field of the function F. The solutions of the system 1 are then the integral curves of the Hamiltonian field ξ_H of the function H. In modern language, a Hamiltonian system is given by a triple $(\mathcal{X}, \omega, H)$, where $\omega \in \Lambda^2\mathcal{X}$ is a closed 2-form on $\mathcal{X}$ having the expression $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$ in some local canonical coordinates.

The system of the form described above is by definition integrable, if there exists a set of n functions $F_i : \mathcal{X} \to \mathbb{R}$ containing H and such that $F_i(q(t), p(t)) \equiv const.$ for every $i = 1, \ldots n$ and for every solution $\gamma(t) = (q(t), p(t)) : \mathbb{R} \to \mathcal{X}$ of the system 1. In addition we demand that F_i is constant along the integral curves of ξ_{F_j} for every $i, j = 1, \ldots, n$. Two functions F_i, F_j with this property are said to Poisson-commute. The functions F_i are called the first integrals of 1.

Clearly, the solutions $\gamma : \mathbb{R} \to \mathcal{X}$ of 1 are confined to the level sets

$$\mathcal{L}_c = \{x \in \mathcal{X} \ ; \ F_i(x) = c_i \ , i = 1, \ldots, n\}$$

of the integrals. When $\mathcal{L}_c$ is compact, Liouville's theorem shows that it is diffeomorphic to the n-dimensional torus T^n. This theorem also establishes the existence of a system $(\phi_1, \ldots, \phi_n)$ of affine coordinates on $\mathcal{L}_c$ such that the solutions γ are linear with respect to these coordinates. Let c be a regular value of the map $\mathcal{F}(x) = (F_1(x), \ldots F_n(x))$ from $\mathcal{X}$ to $\mathbb{R}^n$. Then the fibres of $\mathcal{F}$ close to $\mathcal{L}_c$ are also diffeomorphic to T^n, so locally around $\mathcal{L}_c$ the space $\mathcal{X}$ looks like $T^n \times \mathbb{R}^n$. Choose a basis δ_i of 1-cycles in $H_1(\mathcal{L}_c\ ; \mathbb{Z}) \cong H_1(T^n\ ;\ \mathbb{Z})$ depending smoothly on F_i's and put

$$I_k = \int_{\delta_k} p \cdot dq \quad , \quad k = 1, \ldots, n\,.$$

It is then easily seen that (I, ϕ) are canonical coordinates in which the system 1 has the simple form

$$\begin{aligned} \dot{I}_k &= 0 \\ \dot{\phi}_k &= \tfrac{\partial H}{\partial I_k} \end{aligned} \tag{2}$$

The equations 2 have the obvious solutions $I_k(t) = I_k(0)$, $\phi(t) = t \cdot \frac{\partial H}{\partial I_k} + c_k$. This shows that the concept of integrability described above is reasonable in the sense that it really assures integrability of the system by quadratures.

For a generic conservative dynamical system (without additional preserved quantities) the ergodic theorem tells us that a generic solution $\gamma : \mathbb{R} \to \mathcal{X}$ is dense in the whole isoenergetic hyper-space $H_c = \{x \in \mathcal{X};\ H(x) = c\}$ in $\mathcal{X}$. In the case of an integrable system a generic solution is dense only on its level torus $\mathcal{L}_c$, which is a sub-manifold of codimension n in the $2n$-dimensional space $\mathcal{X}$. This illustrates how special and non-typical the integrable systems are.

Some classical examples of integrable systems

Until the sixties the list of known integrable dynamical systems was very short if old. The oldest examples probably are the rotating heavy rigid bodies or the tops. The kinematics of a top is described by the change in time of the mutual position of two coordinate systems, one fixed in space and the other fixed in the top, both sharing the same origin. At each point of time there is an element $q \in SO(3)$ sending the space coordinates into those fixed in the top. So the natural space of positions (the configuration space) of a top is the group $SO(3)$ of rotations in $\mathbb{R}^3$. Adding momenta gives us the phase space $\mathcal{X} = T^*SO(3)$. The behaviour of a top is determined by its shape and the position of its baricenter. The shape of the top is encoded in its inertia tensor, and this gives rise to a certain left-invariant metric on $SO(3)$ determined by a symmetric 3×3 matrix A. Clearly restricting to the diagonal matrices causes no loss of generality. The diagonal terms (a_1, a_2, a_3) correspond to the principal axes of our body, while the baricenter is determined by its body coordinates (b_1, b_2, b_3). We note that not all such tops are integrable. For the integrability certain conditions on the parameters (a_i, b_i) have to be satisfied . The well known list of integrable cases, found e.g. in [Ar 2] is:

Euler; 1750: The free top, in the absence of the gravitational field, i.e. $b_i = 0$.

Lagrange; 1788: Axially symmetric top with the baricenter on the axis of symmetry; $a_1 = a_2$, $b_1 = b_2 = 0$.

Kowalevskaya; 1889: A symmetric top with the baricenter outside the axis of symmetry. More precisely: $a_1 = a_2 = 2a_3$, $b_3 = 0$

Goryachev-Chaplygin; 1900: Similar to Kovalevskaya's case. $a_1 = a_2 = 4a_3$, $b_3 = 0$.

To summarize, the heavy top is the dynamical systems on the space $\mathcal{X} = T^*SO(3)$ with Hamiltonians of the form $H(q,p) = \langle Ap, p\rangle + V_{(a,b)}(q)$. This system is integrable for the pairs $(A, V_{(a,b)})$ of the metrics and the potentials corresponding to the cases listed above. Among these the most interesting one is Kowalewskaya's top which to this day serves as the inspiration for a considerable body of research.

The other two classical examples we mention come from a different mechanical motivation. The first is the geodesic motion of a particle on an ellipsoid, that was solved by Jacobi in 1838. The other is the motion of a particle under the influence of the linear force (harmonic motion) restricted to the sphere. This was integrated by Carl Neumann in 1859. These two examples turned out to be very influential in the development of modern theory of integrable systems, as we are going to see below. The first system is given by the Hamiltonian $H(q,p) = \|p\|^2$ on the phase space $T^*\mathcal{E}$, where $\mathcal{E}$ denotes the ellipsoid, while the second has the Hamiltonian $H(q,p) = \|p\|^2 + \langle Aq, q\rangle$ on T^*S^n, where A is a symmetric $(n+1) \times (n+1)$ matrix. It is shown in [A-vM 1], that these two systems belong to the same family in the sense, that the integrals of one are obtainable from the integrals of the other.

Of a more recent origin is the third class of examples known under the joint name of Toda lattices. They describe the evolution of some system of particles on the real line interacting among themselves pairwise with exponential forces and different configurations of pairs. Toda lattices were instrumental in the reviving of the interest for the integrable systems. In the sixties it was observed, that the important PDE, the Korteweg-de Vries equation can be viewed as an evolution system with the infinitely many conserved quantities, a situation analogous to Liouville integrability outlined above. Later a close relationship between the Korteweg-de Vries equation and the Toda systems was established by Adler , van Moerbeke, and Kostant. The phase space of a Toda lattice is $T^*\mathbb{R}^n$, and the Hamiltonian in the simplest case is $H(q,p) = \|p\|^2 + \sum_{i=1}^n \exp(q_i - q_{i+1})$, where $q_1 = q_n$.

Close relatives of the Toda lattices are the n-body problems on a line appearing in the work of Moser, Calogero, Sutherland, Yang, Olshanetsky, Perelomov and others. In the basic versions of these systems all the pairs of particles interact, not just the adjacent ones. Restrict the configuration of the particles by the constraint $\sum_{i=1}^n q_i = 0$. Permuting the particles obviously preserves such systems, so their configuration space is $\mathbb{R}^{n-1}/\mathfrak{S}_n$, where $\mathfrak{S}_n$ is the permutation group on n elements acting on the coordinates of points in $\mathbb{R}^{(n-1)} \subset \mathbb{R}^n$. The space $\mathbb{R}^{n-1}$ can be thought of as the maximal torus sub-algebra $\mathfrak{t}$ of Lie algebra $\mathfrak{s}u(n)$ and the group $\mathfrak{S}_n$ as its Weyl group. The quotient $\mathfrak{t}/W_n$ is then the Weyl chamber $\mathcal{C}_{\mathfrak{s}u(n)}$. So these systems can be viewed as describing the motion of a particle in the Weyl chamber $\mathcal{C}_{\mathfrak{s}u(n)}$ under the influence of a certain potential force. On the common phase space $T^*\mathcal{C}_{\mathfrak{s}u(n)}$ the Hamiltonians are of the form

$$H(q,p) = \|p\|^2 + \sum_{i<j} V(q_i - q_j)$$

for certain functions V, e.g. $V(q) = q^{-2}$, $V(q) = \sin^{-2}(q)$, or $V(q) = \delta(q)$.

Let Δ^+ be the system of positive roots of $\mathfrak{su}(n)$ in $\mathfrak{t} \cong \mathbb{R}^{n-1}$ with respect to some ordering, and let $q = (q_1, \dots q_n) \in \mathfrak{t}$. Adjusting the indexation of q_i with the ordering of the roots, we get $(q_i - q_j) = \langle q, \alpha_k \rangle$ for some $\alpha_k \in \Delta^+$. So the above Hamiltonians can be rewritten in the form

$$H(q,p) = \|p\|^2 + \sum_{\alpha_k \in \Delta^+} V(\langle q, \alpha_k \rangle) \tag{3}$$

for the appropriate choice of the function V. In this form the Hamiltonian makes sense for any semi-simple Lie algebra $\mathfrak{g}$, thus giving a family of systems on the cotangent bundles of Weyl chambers $T^*\mathcal{C}_{\mathfrak{g}}$. Integrability of these and related systems was proved by the authors mentioned above. We note, that the Hamiltonians of the Toda lattices can be expressed in the form analogous to 3. The only adjustment that has to be made, is to replace Δ^+ with a system of simple roots $\mathcal{S} \subset \Delta^+$.

There is a common feature in the classical examples mentioned. Their phase spaces are all related to Lie groups in one way or another. In the case of the tops this is $T^*SO(3)$; the C. Neumann problem is happening on T^*S^n, and S^n is the homogeneous space $SO(n+1)/SO(n)$, and as mentioned, the elliptic motion on the ellipsoid is closely related to the C. Neumann' s case. For the n-body problems on the line the phase space $T^*\mathcal{C}_{\mathfrak{g}}$ is flat but still essentially connected to Lie groups. This is not surprising. Lie groups can be thought of as a mathematical formalisation of the notion of symmetry in physics, and clearly the more symmetrical the physical situation is the more it is tractable. In particular, it is easier to decide which Hamiltonians yield integrable systems in the phase space possess a certain amount of symmetry.

There seem to be very few known examples of integrable systems obtained by the classical methods whose configuration and phase space are non-homogeneous. Some can be found in the paper [P-S] and in the references therein. In particular they prove that the geodesic motion on the connected sum $\mathbb{CP}^n \# \ldots \# \mathbb{CP}^n$ is completely integrable.

Hitchin's systems

There is however a large class of known integrable systems on non-homogeneous spaces. These were discovered around 1984 by Hitchin while studying the moduli spaces of solutions of the equation

$$F_A + [\Phi, \Phi^*] = 0\,, \tag{4}$$

where A is a connection on a vector bundle $\mathcal{E} \to C$ over the complex curve C, F_A its curvature, and Φ a section in the bundle $End(\mathcal{E}) \to C$. Hitchin's equation is a two dimensional reduction of the anti-self-dual Yang-Mills equation. In his paper [Hi 2] Hitchin describes the integrable system in the case where $\mathcal{E}$ is a rank two bundle. Subsequently in [Hi 1] he generalises the construction, replacing $\mathcal{E}$ by a principal bundle $\mathcal{P}$ having arbitrary classical Lie group as the structure group. The space $\mathcal{M}_H$ contains the cotangent bundle $T^*\mathcal{M}$ as a dense open subset. In [Hi 1] Hitchin works on this subspace.

As it is well known, the moduli spaces $\mathcal{M}$ of stable holomorphic structures on $\mathcal{P}^{\mathbb{C}} \to C$ are isomorphic to the moduli spaces of flat connections on $\mathcal{P} \to C$. A flat connection in turn yields a representation of the fundamental group $\pi_1(C)$ in the structure group G of $\mathcal{P}$. So Hitchin's construction gives an integrable system on the cotangent bundle $T^*(Hom(\pi_1(C), G)/G)$ for every complex curve C and every compact semi-simple real Lie group G. We note that, even though the spaces $\mathcal{M}$ constitute a family far richer and more diverse than the homogeneous spaces, Lie groups still play a prominent and essential role.

Hitchin's systems have an unusual genesis. Traditionally people started with a certain mechanical system and then tried to integrate it, or at least to prove it is integrable. In the case of the Hitchin's systems though, what is known is that there is an integrable system. More precisely, a system of $n = dim\mathcal{M}$ Poisson-commuting functions on $T^*\mathcal{M}$ is given. What is still not known is the mechanical content of these systems.

Methods of proving the integrability

For a long time the theory concerned with proving the integrability and integrating systems reflected the diversity of examples of integrable systems. One could almost say that there was a different approach for each example. One notable exception is the Hamilton-Jacobi method of separation of variables. This situation started to change in seventies with the introduction of a more systematic use of Lie theory and algebraic geometry in the field.

Very important constructions are those of the moment map and of symplectic reduction, which apply under mild assumptions to the systems with continuous symmetries. For the system $(\mathcal{X}, H)$ to have a continuous symmetry means that there is a Lie group G acting on $\mathcal{X}$ and preserving H. Each one-parameter subgroup of G then gives rise to a function F_i, which Poisson-commutes with H. These functions are the components of the moment map. There is an action of a certain subgroup G_c of G on the level set $\mathcal{X}_c = \{x \in \mathcal{X}; F_i(x) = c_i\}$. Taking the quotient $\mathcal{W} = \mathcal{X}_c/G_c$ we get the symplecticaly reduced system $(\mathcal{W}, \widetilde{H})$, where $dim\mathcal{W} = dim\mathcal{X} - (dimG + dimG_c)$. This method of reducing the dimension of a mechanical system emerges very naturally in the treatment of the Euler's top, where the conserved quantities apart from the energy are the three components of the momentum, which explains the name moment map. The construction was used by Smale and Arnold and subsequently systematised by Marsden and Weinstein.

Methods using Lie theory for proving the integrability of Hamiltonian systems are numerous and diverse. The strategy common to many of them is the following. Let the phase space $\mathcal{X}$ or some symplectic reduction of $\mathcal{X}$ be embedded in $\mathbb{R}^m$ for some m. Think of $\mathbb{R}^m$ as of the dual Lie algebra $\mathfrak{g}^*$ of some Lie group G, and prove that the solution $\gamma(t)$ of the system lies in the orbit of $\gamma(0) \in \mathfrak{g}^*$ with respect to the coadjoint action of G on $\mathfrak{g}^*$. Coadjoint orbits are equipped with the so called Kostant-Kirillov symplectic forms and their symplectic geometry is well understood. In particular for a compact coadjoint orbit $\mathcal{O}$ there are $(1/2)n = dim\mathcal{O}$ Poisson commuting functions $f_k : \mathcal{O} \to \mathbb{R}$. If the Hamiltonian H of the system $(\mathcal{X}, \omega, H)$ happens to be one of them, the integrability of the system is proved.

The functions f_k were constructed by Mischenko and Fomenko in the following way. Let $\lambda \in \mathfrak{g}^*$ be a fixed element and x an indeterminate. Denote by $(q_1, \dots q_r)$

a system of Ad^*_G-invariant polynomials on $\mathfrak{g}^*$, and consider the "shifted" functions $q_i(\alpha + x \cdot \lambda)$. These can be expanded with respect to x, giving

$$q_i(\alpha) = \sum_{j=0}^{d_i} x^j \cdot f_{(j+\sum_{k=1}^{j-1} d_k)}(\alpha) \,,$$

where $d_i = \deg q_i$.

The algebro-geometric aspects of integrability were addressed by Moser ([Mo 2]), Adler and van Moerbeke ([A-vM 2]), Mumford ([Mu]), and others. The common theme in these contributions is the use of the Lax equation, originating from Peter Lax's approach to the KdV-equation. Let $\mathfrak{M}_k$ be the space of $k \times k$ matrices and let $A, B : \mathbb{R} \to \mathfrak{M}_k \otimes \mathbb{C}[z]$ take the values in the matrix valued polynomials. The strategy is then to rewrite the equations defining the system $(\mathcal{X}, H)$ in the form of the Lax equation

$$\dot{A} = [B, A] \tag{5}$$

for some suitable choice of the pair (A, B). Here $[\cdot, \cdot]$ stands for the Lie bracket in $\mathfrak{M}_k \otimes \mathbb{C}[z]$ defined as a tensor product of the commutator of matrices and multiplication of polynomials. At each $t \in \mathbb{R}$ the eigenvalues of $A(t)$ are the solutions $w_1(t, z), \ldots w_k(t, z)$ of the polynomial equation $Q_t(z, w) = \det(w \cdot I - A(t, z))= 0$, so $w_k(t, z)$ are polynomials of degree k in z. In the case, where $A(t, z)$ is a solution of 5, the polynomials $Q_t(z, w)$ and $w_k(t, z)$ are independent of the time parameter $t \in \mathbb{R}$. This is easily seen. At each fixed z_0 and t_0 the equation 5 tells us that the tangent of the solution curve $A(t, z_0) : \mathbb{R} \to \mathfrak{M}_k$ lies in the tangent space of the adjoint orbit in $\mathfrak{M}$ through $A(t_0, z_0)$, since in general $\frac{d}{dt}|_{t=0} Ad_{g(t)}(a) = [g(0), A]$. A solution $A(t, z_0)$ therefore stays in the adjoint orbit of $A(0, z_0)$. The elements lying in the same orbit all have the same spectrum. Suppose, $z \in \mathbb{CP}^1$. Then the equation

$$Q(z, w) = 0$$

defines a complex curve S, called the spectral curve which is a ramified covering of $\mathbb{CP}^1$. This curve is the preserved quantity of the system $(\mathcal{X}, H)$. More precisely, S is in a natural way a divisor of a certain complex surface M ruled over $\mathbb{CP}^1$. The components $F_i(S)$ of the point $S \in |S|$ are the integrals of motion of the system. The feature that makes this method exceptionally powerful and elegant is the following. Let $\mathcal{L}_c \subset \mathcal{X}$ be the level torus defined by $f_i(x) = c_i$ for $i = 1, \ldots n$ on which the flow of the system linearises, as described by Liouville's theorem. Then $\mathcal{L}_c$ turns out to be a real form of an Abelian variety related to the spectral curve S, e.g. the Jacobian $Jac(S)$, or some Prym variety lying in $Jac(S)$. The system $(\mathcal{X}, H)$ can then at least in principle be integrated in terms of theta functions. The common name for such systems is the algebraic completely integrable systems. Clearly the space of matrices $\mathfrak{M}$ can be replaced by some other Lie algebra.

Jacobi's solution of the geodesic motion on the ellipsoid is in a way an early precursor of the spectral curve method. The key step in applying the Hamilton-Jacobi method of separation of variables is of course a suitable choice of coordinates. In the case of the motion on the ellipsoid these are the so called elliptic coordinates obtained as intersections of the system of confocal quadrics. This places the problem firmly in the realm of algebraic geometry, and the solutions provided by Jacobi are indeed Abelian integrals. C. Neumann's treatment of the harmonic motion on the sphere is similar. In [Mo 1], Moser provided a study of these two and some other problems in terms of the spectral curve. In their papers [A-vM 1] and [A-vM 2] Adler and van Moerbeke expanded the use of the method to the tops (with the exception of the Kowalewskaya's case) and on the Toda lattices, thus covering most of the known integrable systems. However the Lax equation method has a somewhat disturbing feature. Namely, finding a suitable Lax pair (A, B) for a particular problem is a matter of a clever guess rather than of an established recipe. Indeed the method is often called the Lax trick.

Note that the "shifted" invariant functions mentioned above are equivalent to a special case of the spectral curve construction. Expanding $Q(z, w)$ with respect to w gives

$$Q(z, w) = \sum_{i=1}^{r} w^i \cdot q_i(A(z)) \ ,$$

where q_i are the invariant polynomials of the relevant Lie algebra. Taking A to be a Lie algebra valued polynomial of degree one gives the shifted invariants construction.

The interaction of the algebraic geometry and Lie theory is very apparent in the proof of the integrability of Hitchin's systems. This is not surprising since the relevant data C and G of the phase space

$$\mathcal{M}_H \cong Hom(\pi_1(C); G^{\mathbb{C}})/G^{\mathbb{C}} \cong T^*(Hom(\pi_1(C); G)/G) \cup \mathcal{D}$$

are coming from both sources. Moreover, the space $\mathcal{M}$ can be thought of as the space of representations as well as the moduli space of holomorphic structures on a principal $G^{\mathbb{C}}$-bundle $P \to C$. Let $a \in \mathcal{M}$. The standard deformation argument then gives

$$T_a\mathcal{M} \cong H^1_a(Hom(\pi_1(C); G)) \cong H^1_{[a]}(C; adP) \ ,$$

where $[a]$ denotes the holomorphic structure on P corresponding to the representation a. By Serre duality we then have $T^*_{[a]}\mathcal{M} \cong H^0_{[a]}(C; adP^* \otimes K)$, where K is the canonical line bundle of C. So the field Φ from the equation 4 is a section of $adP \otimes K$ holomorphic with respect to $[a]$. Let $(q_1, \ldots, q_r)$ be a basis of the $Ad_{G^{\mathbb{C}}}$-invariant polynomials on $\mathfrak{g}^{\mathbb{C}}$. Then $q_i(\Phi)$ is a holomorphic section of the line bundle $K^{d_i} \to C$, i.e. an element of the vector space $H^0(C;\ K^{d_i})$. Combining the Riemann-Roch theorem and the well known formula $\sum_{i=1}^r (2d_i - 1) = \dim \mathfrak{g}$, gives the equality

$$\dim \bigoplus_{i=1}^{r} H^0(C; K^{d_i}) = \dim \mathcal{M} \ .$$

Let $\{[\alpha_{i,j}]\}$ be a basis of the space $H^0(C;K^{d_i})^* \cong H^1(C;K^{1-d_i})$ for each $i = 1,\dots r$ and let $\alpha_{i,j}$ be 1-forms with values in $K^{(1-d_i)}$ representing the cohomology classes $[\alpha_{i,j}]$. Then we can define $\dim \mathcal{M}$ functions $f_{i,j} : T^*\mathcal{M} \to \mathbb{C}$ by $f_{i,j}(\Phi) = \int_C \alpha_{i,j} \cdot q_i(\Phi)$. Let $\mathcal{A}$ denote the space of the differential operators $\bar{\partial} + a : \Omega^0(C; adP) \longrightarrow \Omega^{(0,1)}(C; adP)$ acting by $(\bar{\partial} + a)s = \bar{\partial}s + [a,s]$. The gauge group $\mathcal{G} = \mathcal{C}^\infty(C; G^{\mathbb{C}})$ acts on $\mathcal{A}$ and $\mathcal{M} = \mathcal{A}/\mathcal{G}$. To prove the Poisson commutativity of the functions $\{f_{i,j}\}$ Hitchin used the fact that the space $T^*\mathcal{M}$ is the symplectic quotient of the space $T^*\mathcal{A}$ with respect to the lifted action of $\mathcal{G}$ on $T^*\mathcal{A}$. The Poisson commutation of the functions $\widetilde{f}_{i,j}$ on $T^*\mathcal{A}$ which descend on $f_{i,j}$ under the symplectic quotient in immediate.

Despite the absence of the Lax equation in this situation, one can nevertheless construct the spectral curve S of the system. Again a certain surface M ruled over C is needed, to be precise $M = \mathbb{P}(K^{d_r} \oplus \mathbb{C})$. Let z be a coordinate in C and w a coordinate in the fibre direction of M. Then the zero divisor S of the section

$$\mathcal{Q}(z,w) = w^{d_r} + \sum_{i=1}^{r} w^{(d_r - d_i)} \cdot q_i(z)$$

obviously carries the same information as the functions $f_{i,j}$. Moreover, the flows of Hamiltonian systems $(T^*\mathcal{M}, \omega, f_{i,j})$ are linearised on the $Jac(S)$ when the structure group $G^{\mathbb{C}} = GL(n,\mathbb{C})$ and on some Prym variety lying in $Jac(S)$ when $G^{\mathbb{C}}$ is some other classical Lie group. Hence Hitchin's system is a large family of algebraic integrable systems.

Lax pairs mentioned above can be seen as a special case of Hitchin's systems. Polynomials A of degree d with values in a Lie algebra $\mathfrak{g}$ can be viewed as the holomorphic sections of the bundle $\mathcal{O}(d) \otimes \mathfrak{g} \to \mathbb{CP}^1$. So, if the role of the basic curve C is taken by $\mathbb{CP}^1$ equipped by a divisor of marked points, the moduli spaces of holomorphic bundles are replaced by the moduli spaces of parabolic bundles and the polynomials A become objects analogous to the sections Φ appearing in the Hitchin's systems.

0.2

The main part of this thesis is Chapter 3. Let M be an arbitrary Riemannian symmetric space. Then M is homogeneous, more precisely, there exists a real semi-simple Lie group G and a subgroup U such that $M = G/U$. Moreover, the projection $G \to M$ is a Riemannian submersion. This means that the natural Riemannian metric on M is induced by the Killing form $\mathcal{K}$ on G. Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g} = Lie(G)$ and let $\widetilde{\mathfrak{g}}$ be a real form dual to $\mathfrak{g}$. Denote by β an arbitrary element in $\widetilde{\mathfrak{g}}$ and by $\overline{\beta}$ its conjugate with respect to the real structure corresponding to $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$. Chapter 3 is devoted to the proof and discussion of the following theorem.

Theorem 1 *Let M be an arbitrary symmetric space and T^*M the cotangent bundle over it, equipped with the canonical symplectic structure ω_{can}. Let the Hamiltonian H be given by the formula*

$$H(q,p) = \|p\|^2 + \mathcal{K}(Ad_q(\beta), \widetilde{\beta}) \, .$$

Then the system (T^*M, ω_{can}, H) is an integrable Hamiltonian system.

The list of Riemannian symmetric spaces composed by E. Cartan is long and diverse, so the above theorem gives a large family of integrable systems. This family contains certain known classical examples, but also many new ones. Some of them are discussed in the section 3.5.

The unit sphere $S^n \in \mathbb{R}^{n+1}$ is perhaps the fundamental example of a symmetric space. At the beginning of section 3.5, we show that in case where $M = S^n$ the system (T^*M, ω_{can}, H) from the above theorem coincides with the classical C. Neumann system describing the motion of a particle in a quadratic potential. As we shall see in the chapter 3, symmetric spaces are geodesic sub-manifolds of real Lie groups. The inversion in a semi-simple real group is an anti-linear operation. This is immediately clear for example for compact real groups. Every such group is embeddable in $SU(n)$ for large enough n, and inversion in $SU(n)$ is given by $\alpha^{-1} = \alpha^*$. Therefore the potential $V_\beta(q) = \mathcal{K}(Ad_q(\beta), \widetilde{\beta})$ appearing in the Hamiltonian H is quadratic in some sensible coordinates, and the systems (T^*M, ω_{can}, H) can be viewed as generalisation of the C. Neumann's system to the arbitrary Riemannian symmetric space.

Another obvious family of symmetric spaces are the compact real Lie groups. The "smallest" non-Abelian among them is the group of rotations $SO(3)$. The Hamiltonian $H_{(p)} = \|p\|^2 + V_\beta(q)$ of the system $(T^*SO(3), \omega_{can}, H)$ is invariant with respect to the action of $Stab\ (\beta) \cong U(1)$. The symplectic reduction $(\mu^{-1}(0)/U(1), \widetilde{\omega}, \widetilde{H})$ turns out to be the spherical pendulum $(T^*S^2, \omega_{can}, H_{(p)})$, a classical mechanical system studied already by Huygens and more recently by Duistermaat.

The other systems that we consider in some detail are those whose configuration spaces are the projective spaces $\mathbb{RP}^n$, $\mathbb{CP}^n$ and $\mathbb{HP}^n$. The projective spaces are quotients of spheres. Concretely $\mathbb{RP}^n = S^n/S^0$, $\mathbb{CP}^n = S^{(2n+1)}/S^1$, and $\mathbb{HP}^n = S^{(4n+3)}/S^3$. We describe the systems (T^*M, ω_{can}, H), where $M = \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n$ in terms of the systems on spheres, which descend on (T^*M, ω_{can}, H) when taking the appropriate quotient. This gives a family of new integrable systems describing the motion of a particle confined to a sphere under the influence of certain quartic potentials. More precisely, the following systems are integrable

(i) $(T^*S^n, \omega_{can}, H_{(4)})$, for

$$H_{(4)}(q,p) = \|p\|^2 + 4\langle \beta q, \beta q\rangle - 4\langle \beta q, q\rangle^2 \, ,$$

where $q = (q_1, \dots q_{(n+1)}) \in \mathbb{R}^{(n+1)}$, $\sum q_1^2 = 1$, and $\beta \in \mathfrak{so}(n+1)$.

(ii) $(T^*S^{2n+1}, \omega_{can}, H_{(c4)})$ for

$$H_{(c4)} = \|p\|^2 + 4\langle \beta q, \beta q\rangle - |\langle \beta q, q\rangle|^2 \, ,$$

where $q = (q_1, \dots q_{(n+1)}) \in \mathbb{C}^{(n+1)}$, $\sum |q_i|^2 = 1$, and $\beta \in \mathfrak{su}(n+1)$.

(iii) $(T^*S^{(4n+3)}, \omega_{can}, H_{(q4)})$ for

$$H_{(q4)} = \|p\|^2 + 4\langle \beta q, \beta q\rangle - |\langle \beta q, q\rangle|^2 \, ,$$

where $q = (q_1, \dots q_{(n+1)}) \in \mathbb{H}^{(n+1)}$, $\sum |q_i|^2 = 1$ and $\beta \in \mathfrak{sp}(n+1)$.

The main difference between the above systems is the degree of invariance of their Hamiltonians. They are all invariant with respect to the liftings of the actions corresponding to the appropriate Hopf fibrations. The relevant groups acting are: $\mathbb{Z}_2$ in the case *(i)*, the group $U(1)$ in the case *(ii)*, and $SU(2)$ in the case *(iii)*. The integrability of the system *(ii)* follows from that of the system $(T^*\mathbb{CP}^n, \omega_{can}, H)$ using the lemma 1, and the integrability of *(iii)* is the consequence of *(ii)* being integrable.

The invariance of *(ii)* and *(iii)* enables us to apply the symplectic quotient and thus obtain new integrable systems. Let $\mu : T^*S^{(2n+1)} \to i \cdot \mathbb{R}$ be the moment map of the $U(1)$-action on *(ii)*. The symplectic quotient $\mu^{-1}(0)/U(1)$ brings us back to the system $(T^*\mathbb{CP}^n, \omega_{can}, H)$, but taking a non-zero value $\gamma \in i \cdot \mathbb{R}$ changes the system. The phase space remains diffeomorphicaly the same, and essentially so does the Hamiltonian. What changes is the symplectic structure. Instead of ω_{can} we get the structure $\omega_{can} + \alpha_\gamma$, and the 2-form α_γ is called the magnetic term. In our case we get

$$(T^*\mathbb{CP}^n \ , \ \omega_{can} + \gamma\omega^*_{(FS)} \ , \ H_{(m)}) \, ,$$

where $\omega^*_{(FS)}$ is the pull-back of the Fubini-Study form to $T^*\mathbb{CP}^n$. Mechanically such a perturbation of the symplectic form means adding of the action of some magnetic force to the system. (This topic is discussed e.g. in [Ma].) In particular, taking $\mathbb{CP}^1 = S^2$ gives us a system describing the motion of a particle confined to the sphere S^2 under the influence of a quadratic potential force, and a force of a magnetic monopole placed in the centre of the sphere.

The case *(iii)* is slightly different, since the group $SU(2)$ acting on it is non-Abelian. The symplectic quotient $\mathcal{N} = \mu^{-1}(\gamma)/U(1)_\gamma$ is a $\mathbb{CP}^1$-fibration over $T^*\mathbb{HP}^n$. The fibre $\mathbb{CP}^1$ is the part of the phase space parametrising the internal structure of a particle moving in a Yang-Mills force. The space $\mathcal{N}$ can also be thought of as a sub-bundle of $T^*\mathbb{CP}^{(2n+1)}$, equipped with the symplectic structure $\widetilde{\omega}_{can} + \omega^*_{(FS)}$. So the system we get is

$$(\mathcal{N} \ , \ (\omega_{can} + \omega^*_{(FS)})_{/\mathcal{N}} \ , \ H_{(YM)}) \, ,$$

where $H_{(YM)}$ is again essentially the same as in the system $(T^*\mathbb{HP}^n, \omega_{can}, H)$. In case when we start with the system $(T^*S^7, \omega_{can}, H)$ the above construction describes the motion of a particle on the sphere S^4 influenced by a quadratic potential and the Yang-Mills field.

Proceeding in the same way as in the case of the system *(ii)*, the symplectic reduction of $(T^*SO(3), \omega_{can}, H)$ with respect to a non-zero element $\gamma \in i \cdot \mathbb{R}$, gives the system $(T^*S^2, \omega_{can} + \gamma\omega_{(FS)}, H_{(pm)})$, which represents a spherical pendulum moving in a magnetic force.

In general we can apply these observations to all the systems (T^*G, ω_{can}, H), where G is a compact real group, and $H(q,p) = \|p\|^2 + \mathcal{K}(Ad_q(\beta), \beta)$. The stabiliser T_β of β acts on the system, so we can form different symplectic quotients. Taking $0 \in \mathfrak{g}$ as the value of the the moment map $\mu : T^*G \to \mathfrak{t}_\beta^*$ gives the system $(T^*\mathcal{O}_\beta, \omega_{can}, \widetilde{H})$, describing the motion on the coadjoint orbit $\mathcal{O}_\beta$ of β in a quadratic potential, while taking $0 \neq \gamma \in \mathfrak{t}_\beta^*$ yields $(T^*\mathcal{O}_\beta,\ \omega_{can} + \alpha_\gamma,\ \widetilde{H}_\gamma)$, adding the influence of a magnetic field to the motion.

The source for the systems (T^*M, ω_{can}, H) are Nahm's equations

$$\dot{T}_i + \frac{1}{2}\sum \epsilon_{i,j,k}[T_j, T_k] \quad i = 1, 2, 3\,, \tag{6}$$

where $T_i : I \to \mathfrak{g}$ are functions with values in some semi-simple Lie algebra. These equations originate in the gauge theory, being a rewriting of the Bogomolny equations for the magnetic monopoles in $\mathbb{R}^3$. (See [Nahm], [Do], [Hi 3]) It was observed by Donaldson in [Do], that in the case, where $\mathfrak{g} = \mathfrak{s}u(2)$, Nahm's equations give rise to the variational problem, describing the motion of a particle in the hyperbolic 3-space $\mathcal{H}^3$ under the influence of the potential $V_\beta(h) = Tr(Ad_h(\beta) \cdot \beta^*)$. More precisely, the solutions of 6 are in one-to-one correspondence with the solutions of the variational problem on $\mathcal{H}^3$. In the section 3.1.2 this statement is generalised to the arbitrary semi-simple Lie algebra $\mathfrak{g}$ giving the variational systems on any homogeneous space of the form $\mathcal{H} = G^{\mathbb{C}}/G$, where $\mathfrak{g} = Lie(G)$, and $G^{\mathbb{C}}$ is the comlexification of G. From this we get the systems on any symmetric space M, since we can find a G that $M \subset \mathcal{H}$ will be a fixed point set of an involution and thus a totally geodesic sub-manifold. This is shown in 3.1.1. The precise relationship between Nahm's equations and the systems (T^*M, ω_{can}, H) is given in proposition 24 in 3.1.2. It states that the solutions of (T^*M, ω_{can}, H) are in one-to-one correspondence with the solutions of the equations 6, where

$$T_1, T_3 : I \longrightarrow i\mathfrak{p} \quad , \quad T_2 : I \longrightarrow \mathfrak{u}\,.$$

Here $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ corresponding to the symmetric space $M = G/U$, and $\mathfrak{u} = Lie(U)$.

Nahm's equations have a rewriting in the form of Lax equation

$$\dot{\Phi}_t(z) = [\frac{d}{dz}(\Phi_t(z))\ ,\ \Phi]\,, \tag{7}$$

where $\Phi_t(z) = (T_2 + iT_3) - z(2iT_1) + z^2(T_2 - T_3)$. This places our systems in the context of algebraic completely integrable systems studied by Adler, van Moerbeke and others. It should be noted that in this case the Lax form is not a mere trick, but has a clear gauge theoretical motivation. To every connection A on the bundle $\mathcal{E} \to \mathbb{R}^4$ which carries a metric there corresponds a unique $\bar{\partial}$-operator $\bar{\partial}_A$ making the bundle $\mathcal{E}$ holomorphic with respect to a chosen complex structure on $\mathbb{R}^4$. The complex structures on $\mathbb{R}^4$ are parametrised by $\mathbb{CP}^1$. The connection A satisfies the ASD equation $*F_A = -F_A$ if and only if $\bar{\partial}_A$ is an integrable holomorphic structure with respect to every complex structure $z \in \mathbb{CP}^1$. Nahm's equations can be thought of as the ASD equations for the connections on $\mathcal{E}$ which are invariant with respect to three of the four directions in $\mathbb{R}^4$. Rewriting them in the above sense gives precisely the equation 7, where the spectral parameter z represents the complex structure on $\mathbb{R}^4$.

In the previous section we have seen that Hitchin's systems constitute the largest known family of integrable systems, so it makes sense to try to establish how the systems (T^*M, ω_{can}, H) fit into it. It is immediately clear from 7, that the base curve C will be $\mathbb{CP}^1$ with a certain divisor D of marked points. In the subsection 3.2.3 we get the following result. Let $\mathcal{M}_D$ be the moduli space of the framed $G^{\mathbb{C}}$-bundles over $\mathbb{CP}^1$ with the marked points $D = \{p_1, p_2, p_3, p_4\}$. Divide the points p_i into pairs and let the elements of the pairs coalesce, giving the divisor D_d of two double points. Denote the resulting moduli space by $\mathcal{M}_{D^d}$. Then we have:

The system $(T^*G^{\mathbb{C}}, \omega_{can}, H)$ *is Hitchin's system on* $T^*\mathcal{M}_{D^d}$*.* (8)

The above statement enables us to prove that the systems (T^*M, ω_{can}, H) are actually integrable. The integrability of the Nahm's equations is well known, it follows e.g. from 7. The problem we have here, is to prove that the integrals Poisson-commute with respect to the right symplectic structure. This is done in the subsection 3.3.1.

The systems on T^*M are obtainable from the one above by imposing appropriate involutions. Since the dimension of the problem diminishes, some of the integrals of the "master system" become constant on the whole phase space. In other words they assume the role of constraints. These constraints are constructed in the section 3.4, and interpreted in terms of the spectral curve at the end of the section. This description should hopefully be useful in the explicit integration of the systems in terms of theta functions.

The occurrence of relatives of Hitchin's systems in statement 8 motivated Chapter 2. There Hitchin's systems on the moduli spaces of parabolic bundles are studied. That such systems should exist is not surprising. Roughly speaking, a moduli space of parabolic bundles over a curve with (say) one marked point is a fibre bundle having the moduli space of the appropriate holomorphic bundles as the base and a coadjoint orbit of the structure group as the fibre. Both these spaces are configuration spaces of certain integrable systems, so it is reasonable to hope for the global existence of the right number of Poisson-commuting functions on the total space. Moreover, such systems were already constructed by E. Markman in [Mk]. For the special case, where the base curve is the projective line $\mathbb{CP}^1$ with marked points, this was previously done by Beauville in [Be]. The approach of Beauville and Markman is an algebro-geometric one. We follow Hitchin's approach which is motivated by the gauge-theoretic source of the construction and therefore of a more differential-geometric nature. The advantage of this approach is, that it gives us a more detailed description of the symplectic structures on the spaces involved. As a result we obtain two essentially different families of integrable systems, as we outline below.

First in the section 2.1 we construct the cotangent bundle $T^*\mathcal{M}_D$ over the moduli space of $G^{\mathbb{C}}$-bundles $P \to C$ framed at the divisor D of marked points. Following Hitchin, we describe $T^*\mathcal{M}_D$ as the symplectic quotient of $T^*\mathcal{A}$, where $\mathcal{A}$ is the space of $\bar{\partial}$-operators on P, with respect to the action of the subgroup $\mathcal{G}_D = \{g \in \mathcal{G};\ g(p_i) = id, p_i \in D\}$ of the gauge group $\mathcal{G}$. There is the obvious action of the group $G_D^{\mathbb{C}} = \prod_{i=1}^{deg(D)} G_i^{\mathbb{C}} \cong \mathcal{G}/\mathcal{G}_D$ on $T^*\mathcal{M}_D$. Let $B_D = \prod_{i=1}^{deg(D)} B_i \subset G_D^{\mathbb{C}}$ be a subgroup such that $B_i \subset G_i^{\mathbb{C}}$ is a Borel subgroup for every i. Than we can form two different symplectic quotients of the space $T^*\mathcal{M}_D$.

(i) $(T^*\mathcal{M}_{par}\ ,\ \omega_{can}) = \mu_B^{-1}(0)/B_D$,where μ_B is the moment map corresponding to the action of B_D.

(ii) $((\mathcal{T}^*\mathcal{M})_{par}^{\lambda_D}\ ,\ \omega_{\mathcal{M}KK}) = \mu_D^{-1}(\lambda_D)/H_D$, where μ_D is the moment map of the $G_D^{\mathbb{C}}$-action, $\lambda_D \in Lie(G_D^{\mathbb{C}})$ a regular element, and $H_D \subset G_D^{\mathbb{C}}$ the stabiliser of λ_D.

The detailed description of these objects is given in the section 2.2. The situation can be described in the following way. Let $\mathcal{S}' \subset (\bigoplus_{i=1}^{deg(D)} \mathfrak{h}_i)^* = (Lie(H_D))^*$ consist of regular elements λ_D and let $\mathcal{S} = \mathcal{S}' \cup \{0\}$ where $\{0\} = (0, \ldots, 0) \in (Lie(H_D)^*$. We show that the spaces $(\mathcal{T}^*\mathcal{M})_{par}^{\lambda_D}$ and $T^*\mathcal{M}_{par}$ are diffeomorphic. So we have a family of symplectic structures ω_{λ_D} on $T^*\mathcal{M}_{par}$ parametrised by $\mathcal{S}$ where $\omega_{\lambda_D} = \omega_{\mathcal{M}KK}$ for regular λ_D and $\omega_{\lambda_D} = \omega_{can}$ for $\lambda_D = 0$. The spaces $(T^*\mathcal{M}_{par}, \omega_{\lambda_D})$ are holomorphic symplectic spaces, that is for every λ_D there is a complex structure I_{λ_D} on $T^*\mathcal{M}_{par}$ such that ω_{λ_D} is holomorphic with respect to I_{λ_D}. The space $(T^*\mathcal{M}_{par}, \omega_{\lambda_D})$ is an exceptional member of our family in the sense described in section 2.3. In section 2.3 we construct integrable systems for every member of the family $\{(T^*\mathcal{M}_{par}, \omega_{\lambda_D})\ ;\ \lambda_D \in \mathcal{S}\}$. The Poisson commuting integrals are induced from the Poisson commuting functions previously defined on $T^*\mathcal{M}_D$. In section 2.4 we describe the spectral curves and the corresponding Abelian varieties of these integrable systems for the case where $G^{\mathbb{C}} = SL(n;\mathbb{C})$. The spectral curve S of the system on $(T^*\mathcal{M}_{par}, \omega_{\lambda_D})$ lies in the projectivizaton of the line bundle $K(D) \to C$. We find that the intersections of S

with the fibres $K(D)_{p_i}$ above the points $p_i \in D$ are fixed and determined by λ_D. The exceptionality of the system on $(T^*\mathcal{M}_{par}, \omega_{can})$ is reflected by the fact that in this case the intersections $S \cup K(D)_{p_i}$ are contained in the zero section of $K(D) \to C$. So the curve S has ramification points of maximal degree at $p_i \in D$. Finally we describe how the parabolic structure on $E = \pi_*(L) \to C$ is recovered from the line bundle $L \to S$ on the spectral curve in the cases, where $E \in (\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}$ and $E \in T^*\mathcal{M}_{par}$.

In section 1.2 we study real symplectic structures on complex coadjoint orbits. The situation we address is the following. Let $G^{\mathbb{C}}$ be a complex Lie group, $(\mathfrak{g}^{\mathbb{C}})^*$ its dual Lie algebra and $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ a coadjoint orbit of a generic element $\lambda_0 \in (\mathfrak{g}^{\mathbb{C}})^*$. Every coadjoint orbit $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ carries a natural symplectic structure ω_{KK} called the Kostant-Kirillov structure. One way of obtaining the symplectic space $(\mathcal{O}^{\mathbb{C}}_{\lambda_0}, \omega_{KK})$ is taking the symplectic quotient $\mu^{-1}(\lambda_0)/H_{\lambda_0}$, where $\mu : T^*G^{\mathbb{C}} \to (\mathfrak{g}^{\mathbb{C}})^*$ is the moment map of the $G^{\mathbb{C}}$-action on $T^*G^{\mathbb{C}}$ and $H_{\lambda_0} \subset G^{\mathbb{C}}$ is the stabiliser of λ_0 with respect to the coadjoint action. On the other hand, one can form the symplectic quotient $\mu_B^{-1}(0)/B = T^*G^{\mathbb{C}}/B$, where $B \subset G^{\mathbb{C}}$ is a Borel subgroup and μ_B the appropriate moment map. In both cases we treat the space $T^*G^{\mathbb{C}}$ as a real symplectic manifold and forget about the complex structure it carries. The two quotients are diffeomorphic, but obviously not symplectomorphic. In Proposition 3 we describe the relationship between the two. Again, there is a diffeomorphism P and a 2-form β, such that

$$P : (\mathcal{O}^{\mathbb{C}}_{\lambda_0} \ , \ \omega_{KK}) \longrightarrow (T^*(G^{\mathbb{C}}/B) \ , \ \omega_{can} + \beta)$$

is a symplectomorphism. The form β is the magnetic term. An explicit formula for β is given. Magnetic term is discussed by many authors. The situation usually considered is the difference between two symplectic quotients of the form $\mu^{-1}(a)/H_a$ and $\mu^{-1}(b)/H_b$, where $\mu : M \to \mathfrak{g}$ is the moment map of the G-action on M. In the case when G is not Abelian the spaces $\mu^{-1}(a)/H_a$ and $\mu^{-1}(b)/H_b$ are not necessarily diffeomorphic. In our case we consider two diffeomorphic quotients, but the groups acting are different. Our approach follows the one of Duval, Elhadad and Tuynman in [D-E-T]. In the Proposition 4 we generalise 3 to the quotients of the symplectic spaces of the form T^*P, where $P \to B$ is a principal $G^{\mathbb{C}}$-bundle.

Chapter 1

Some symplectic geometry

In the first section we briefly recall a few basic definitions of symplectic geometry. In particular we will define integrable systems and describe symplectic reduction which will be an important tool in the sequel. The second section is devoted to a more detailed study of a certain aspect of the symplectic geometry of complex coadjoint orbits. We will compare two essentially different symplectic structures on these orbits.

1.1 Preliminaries

We begin by recalling the definition of the fundamental object of our interest.

Definition 1 *Let M be a $2n$-dimensional complex manifold and $\omega \in \Omega^2(M)$ a closed non-degenerate 2–form, i.e. $d\omega = 0$ and $\omega^n \neq 0$. Then ω is called a symplectic structure on M and (M, ω) is called a symplectic manifold.*

Let $H : M \longrightarrow \mathbb{R}$ be a function. The vector field X_H on M satisfying the equation

$$i(X_H)\omega = dH$$

is called a Hamiltonian vector field.

A triple (M, ω, H), where (M, ω) is a symplectic manifold and H is a function on M is called a Hamiltonian dynamical system. The solutions of such a system are the curves $\gamma : I \longrightarrow M$ satisfying the equation

$$\dot{\gamma} = X_H \tag{1.1}$$

Remark 1 *The Darboux theorem ensures the existence of a system of local coordinates $(p_1, ..., p_n, q_1, ..., q_n)$ on M such that $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$. The equation 1.1* *rewritten in these coordinates becomes the classical Hamiltonian system*

$$\dot{q} = \partial H/\partial p$$

$$\dot{p} = -\partial H/\partial q.$$

Let G be a Lie group acting symplectically on M. That means that for every diffeomorphism $\widetilde{g} : M \longrightarrow M$ given by $\widetilde{g}(m) = g \cdot m$, we have

$$\widetilde{g}^*(\omega) = \omega.$$

Choose an element ξ in the Lie algebra $\mathfrak{g}$ of the Lie group G. Then we can define a vector field X_ξ by

$$X_\xi(m) = \frac{d}{dt} |_{t=0} \,(exp(t\xi))(m).$$

Locally there always exists a function $f : M \longrightarrow \mathbb{C}$ such that

$$df = i(X_\xi)\omega.$$

Under the assumption $H^1_{DR}(M) = 0$ such a function exists globally and it is unique up to an additive constant.

Choose now a basis $(\xi_1, ..., \xi_n)$ for $\mathfrak{g}$ and define a map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

by

$$\mu(m) = \sum_{i=1}^{n} f_i(m)\xi^i.$$

Here f_i are Hamiltonian functions of the vector fields X_{ξ_i} and $\{\xi^i\} \in \mathfrak{g}^*$ is the basis dual to $\{\xi_i\}$. Assume that the mapping μ has the following equivariance property:

$$\mu(g \cdot m) = Ad^*_g \mu(m).$$

The mapping $\mu : M \longrightarrow \mathfrak{g}^*$ is called the momentum mapping for our G-action on M.

Theorem 2 (Symplectic-Reduction) *Let $\alpha \in \mathfrak{g}^*$ and let G_α be the isotropy group of α with respect to the coadjoint action of G on $\mathfrak{g}$. Then the manifold*

$$\mu^{-1}(\alpha)/G_\alpha$$

is a symplectic manifold with the symplectic structure $\widetilde{\omega}$ for which we have

$$i^*\omega = \pi^*\widetilde{\omega}.$$

Here $i : \mu^{-1}(\alpha) \longrightarrow M$ is the inclusion and $\pi : \mu^{-1}(\alpha) \longrightarrow \mu^{-1}(\alpha)/G_\alpha$ the natural projection.

The space $\mu^{-1}(\alpha)/G_\alpha$ is often called the Marsden-Weinstein reduction of the symplectic G-space M.

Definition 2 *Let $f, g : M \longrightarrow \mathbb{C}$ be two functions. Their Poisson bracket is defined by*

$$\{f, g\} = \langle df, X_g \rangle = \omega(X_f, X_g) = -\langle dg, X_f \rangle$$

Later we will use the following proposition proved by Hitchin in [Hi 1].

Proposition 1 *Let M be a symplectic G-space and let f, g be two G-invariant functions on M. If f and g Poisson commute on M, i.e. $\{f, g\} = 0$, then $\widetilde{f}$ and $\widetilde{g}$ Poisson commute on $\mu^{-1}(\alpha)/G_\alpha$. Here $\widetilde{f}$ and $\widetilde{g}$ are the functions on the reduced space $\mu^{-1}(\alpha)/G_\alpha$ naturally induced by the G-invariant functions f and g.*

□

Definition 3 *Let (M^{2n}, ω, H) be a Hamiltonian system. Suppose there exist n functions*

$$H = H_1, ..., H_n$$

such that they pairwise Poisson commute and are functionally independent, i.e. $dH_1 \wedge \ldots \wedge dH_n \neq 0$ almost everywhere. Then the system (M^{2n}, ω, H) is called integrable.

When referring to systems integrable in the sense defined above the terms complete integrability and Liouville integrability are also frequently used. Suppose the level sets $\mathcal{L}_c = \{x \in M;\ H_i = c_i\}$ are compact. Then by Liouville's theorem they are diffeomorphic to a disjoint union of n-dimensional tori diffeomorphic to T. If we have an integrable system we can define a symplectic action of the Abelian group T almost everywhere on M. This action is given by the following prescription. Let $t = (t_1, \ldots, t_n) \in T$, where the torus is represented as the quotient $T = \mathbb{R}^n/\Lambda$ for some lattice $\Lambda \cong \mathbb{Z}^n \subset \mathbb{R}^n$, and t_i are the coordinates of t with respect to Λ. Let $H_t = t_1 H_i + \ldots + t_n H_n$ be a new Hamiltonian and γ_t its integral curve such that $\gamma_t(0) = m$. Then we set:

$$t \cdot m = \gamma_t(1)\ .$$

By its construction this action is symplectic. In addition the Hamiltonian H is T-invariant. The momentum mapping $\mu : M \longrightarrow \mathfrak{t}^*$ is defined by

$$\mu(m) = \sum_{i=1}^{n} H_i(m) \tau^i$$

where $\{\tau^i\}$ is a basis of $\mathfrak{t}^*$. In this case the Marsden-Weinstein quotient $\mu^{-1}(t)/T$ is a discrete set of points for a generic t.

We conclude this subsection with the following lemma.

Lemma 1 *Let (M, ω, H) be a Hamiltonian system and assume that the Hamiltonian H is invariant with respect to the symplectic action of an r-dimensional torus T on n- dimensional space M. Let $N = \mu^{-1}(a)/T$ be the symplectic quotient corresponding to a regular value $a \in \mathfrak{t}^*$ of the moment map μ, and suppose that the Hamiltonian system $(N, \widetilde{\omega}, \widetilde{H})$ is integrable. Then the original system (M, ω, H) is also integrable.*

Proof: Let $\widetilde{H_i}$, $i = 1, \ldots, n-r$ be independent Poisson-commuting integrals of the system $(N, \widetilde{\omega}, \widetilde{H})$. The only problem in proving this lemma is to define the functions $H_i : M \to \mathbb{R}$ which will induce the functions $\widetilde{H}_i$ after descending on the symplectic quotient. Denote by $\pi : M \to P$ the natural projection on the space of T-orbits $P = M/T$, and let $\mu(m) = \sum_{j=1}^{r} F_j(m)\tau^j$ be the expression of the moment map

$$\mu : M \longrightarrow \mathfrak{t}^*$$

given by our torus action with respect to some basis $\{\tau^j\}_{j=1,\ldots,r}$ of $\mathfrak{t}^*$. Clearly the functions F_j, $j = 1, \ldots, r$ are constant on $\mu^{-1}(a)$ and therefore yield the constant functions $\widehat{F}_j$ on $N = \mu^{-1}(a)/T \subset P$. For every Hamiltonian vector field ξ_{F_j} restricted on $\mu^{-1}(a)$ we have $\imath(\xi_{F_j})\omega = 0$, where ω is the symplectic form on M restricted to $\mu^{-1}(a)$. Therefore, as it is proved in [G-S], page 175, the distribution $\Xi \subset T\mu^{-1}(a)$ spanned by the vector fields $(\xi_{F_1}, \ldots, \xi_{F_r})$ is involutive and it integrates into a foliation $\mathcal{F}_\Xi$ of $\mu^{-1}(a)$ which in our case is fibrating over N, i.e. there is a fibre bundle $\mu^{-1}(a) \to N$ whose fibres are the leaves of the foliation $\mathcal{F}_\Xi$. Denote by ∇F_j, $j = 1, \ldots r$ the fields dual to the Hamiltonian fields ξ_{F_j} with respect to the non-degenerate form ω on M. The Darboux theorem for sub-manifolds (see [G-S], page 155) then enables us to write ω locally at the point $\alpha \in \mu^{-1}(a) \subset M$ in the form

$$\omega = \sum_{i=1}^{(n-r)} dq_i \wedge dp_i + \sum_{j=1}^{r} dh_j \wedge dg_j \; ,$$

where the 1-forms dh_j and dg_j dual to the vector fields ξ_{F_j} and ∇F_j respectively. From this we see, that for every $j = 1, \ldots r$

$$\imath(\nabla \widetilde{F}_j)\widetilde{\omega} = 0 \; ,$$

where $\nabla \widetilde{F}_j$ are the vector fields on $P = M/T$ induced by the T-invariant fields ∇F_j, and $\widetilde{\omega}$ is the form on P such that $\pi^*\widetilde{\omega} = \omega$. By the argument cited above, the distribution $\Delta \subset TP$ spanned by $\{\nabla F_i;\, i = 1, \ldots, r\}$ is involutive and integrates into the foliation $\mathcal{G}_\Delta$. This provides us with another fibre bundle $p : P \to N$ having the leaves of $\mathcal{G}_\Delta$ as fibres.

Define now the functions $\widehat{H_j} : P \to \mathbb{R}$ by $\widehat{H}_j(m) = \widetilde{H}_j(p(m))$ for every $j = 1, \ldots, n-r$, and finally the functions $H_j : M \to \mathbb{R}$ by $H_j(n) = \widehat{H}_j(\pi(n))$.

We claim that we can take the functions H_j , $j = 1, \ldots, n-r$, and F_j, $j = 1, \ldots r$ as the system of n independent commuting integrals on the space M. The functions F_j obviously Poisson-commute among themselves. In addition, the functions H_i are constant along the integral curves of the Hamiltonian vector fields ξ_{F_j} since these curves lie in the fibres of the projection π. Therefore we have $\langle dH_i, X_{F_j}\rangle = \{H_i, F_j\} = 0$ for every pair (H_i, F_j). That the extended integrals H_j commute among themselves is clear from their construction, as is the independence of the system $\{H_j, F_i\ ; j = 1, \ldots, r,\ i = n-r, \ldots, n\}$. □

1.2 Symplectic structures on complex coadjoint orbits

Let $G^{\mathbb{C}}$ be a complex semi-simple Lie group and $\mathfrak{g}^{\mathbb{C}}$ its Lie algebra. In this section we will describe some aspects of the symplectic geometry of the homogeneous space $G^{\mathbb{C}}/H$, where H is a Cartan subgroup in $G^{\mathbb{C}}$, i.e. a maximal Abelian subgroup. The spaces $G^{\mathbb{C}}$, H and $G^{\mathbb{C}}/H$ are all complex, but in this section we will ignore their complex structures and will be concerned with the underlying real spaces. If for example $G^{\mathbb{C}} = SL(n;\mathbb{C})$, then we will think of $G^{\mathbb{C}}$ as the subgroup of $SL(2n;\mathbb{R})$ with the embedding

$$a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} .$$

It is well known that the homogeneous space $G^{\mathbb{C}}/H$ is endowed with a family of holomorphic symplectic forms, but the symplectic forms we will be concerned with are real.

Choose a semi-simple regular element λ_0 in the dual Lie algebra $(\mathfrak{g}^{\mathbb{C}})^*$ and suppose it is contained in a compact Cartan subalgebra $\mathfrak{t}$. The stabiliser of λ_0 with respect to the coadjoint action is then a commutative group H_{λ_0} whose Lie algebra is the Cartan sub-algebra $\mathfrak{h}_{\lambda_0} = \mathfrak{t}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ containing λ_0. Denote the coadjoint orbit of λ_0 by $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$. We have then

$$\mathcal{O}^{\mathbb{C}}_{\lambda_0} = G^{\mathbb{C}}/H_{\lambda_0} .$$

In general adjoint and coadjoint orbits are different. But in the case, where the Lie algebra in question is equiped with an ad-invariant non-degenerate scalar product the two types of orbits are diffeomorphic. In our case such scalar product is provided by the real part of the Killing form on $\mathfrak{g}^{\mathbb{C}}$.

Recall that a generic class in the cohomology group $H^2(G^{\mathbb{C}}/H;\mathbb{R})$ is represented by a symplectic form on $G^{\mathbb{C}}/H$, and that in addition to this, the group $H^2(G^{\mathbb{C}}/H;\mathbb{R})$ can be identified with the real Cartan sub-algebra $\mathfrak{t} \subset \mathfrak{g}^{\mathbb{C}}$. These facts can be found in [D-H]. As we shall see, the zero class $[0] \in H^2(G^{\mathbb{C}}/H;\mathbb{R})$ is not generic, but it is nevertheless represented by a symplectic form on $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$. The aim of this section is to provide a comparison between the form corresponding to $[0] \in H^2(G^{\mathbb{C}}/H;\mathbb{R})$ and those corresponding to generic classes in $H^2(G^{\mathbb{C}}/H;\mathbb{R})$.

1.2.1

One of the natural ways of equipping the homogeneous space $G^{\mathbb{C}}/H$ with a symplectic structure in to represent it as a symplectic quotient of another, simpler symplectic space, namely the cotangent bundle $T^*G^{\mathbb{C}}$ equipped with the (real) cotangent symplectic form. We can obtain $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ as a symplectic quotient of $T^*G^{\mathbb{C}}$ in two fundamentally different ways. As a result we get two different symplectic forms on $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$.

Our considerations would be valid with some modifications for more general real Lie groups. The reason we concentrate on the case of a complex Lie group $G^{\mathbb{C}}$ viewed as a real group is that the two different symplectic quotients we will encounter yield difeomorphic but not symplectomorphic spaces. In the case of a general real Lie group the quotients would not be diffeomorphic.

We now describe the two symplectic forms in question.

(i) Kostant-Kirillov symplectic form

Let $G^{\mathbb{C}}$ act on itself, say, by right translations. The lifting of this action on $T^*G^{\mathbb{C}}$ gives a symplectic action whose moment map

$$\mu_1 : T^*G^{\mathbb{C}} \longrightarrow (\mathfrak{g}^{\mathbb{C}})^*$$

is given by the formula

$$\mu_1(g,\lambda) = Ad^*_g(\lambda).$$

(Here we trivialised $T^*G^{\mathbb{C}}$ by left translations.) Since $\mu_1^{-1}(\lambda_0)$ is isomorphic to $G^{\mathbb{C}}$ it is clear that

$$\mu_1^{-1}(\lambda_0)/H_{\lambda_0} \cong G^{\mathbb{C}}/H_{\lambda_0}\,.$$

The canonical symplectic form ω on $T^*G^{\mathbb{C}}$ is exact; it is the derivative of the tautological form α, so we have

$$\omega(X,Y) = d\alpha(X,Y) = Y\cdot\alpha(X) - X\cdot\alpha(Y) + \alpha([X,Y]) \qquad (1.2)$$

for every pair X, Y of vector fields on $T^*G^{\mathbb{C}}$. The tautological form α is given by the formula

$$\alpha_\lambda(X) = \langle \lambda, \pi_*(X)\rangle_{\pi(\lambda)}$$

where $\pi : T^*G^{\mathbb{C}} \longrightarrow G^{\mathbb{C}}$ is the natural projection and $\langle\cdot,\cdot\rangle_{\pi(\lambda)}$ is the pairing of covectors and vectors at the point $\pi(\lambda) \in G^{\mathbb{C}}$. Let us trivialise the cotangent bundle

$T^*G^{\mathbb{C}}$ by the left translations to get the isomorphism $T^*G^{\mathbb{C}} \cong G^{\mathbb{C}} \times (\mathfrak{g}^{\mathbb{C}})^*$. Choose a point $(g, \lambda) \in G^{\mathbb{C}} \times (\mathfrak{g}^{\mathbb{C}})^*$, and two vectors (a_g, α) and (b_g, β) in the tangent space $T_{(g,\lambda)}(G^{\mathbb{C}} \times (\mathfrak{g}^{\mathbb{C}})^*)$, where $a_g, b_g \in T_g G^{\mathbb{C}}$. Denote by $dL_{g^{-1}} : T_g G^{\mathbb{C}} \to T_e G^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}$ the derivative of the left translation by g^{-1}. Then we can deduce from the equation 1.2 the following formula for the symplectic form on $T^*G^{\mathbb{C}}$ expressed in the left trivialisation:

$$\begin{aligned} \omega_{(g,\lambda)}((a_g, \alpha), (b_g, \beta)) &= \langle \alpha \ , \ dL_{g^{-1}}(b_g) \rangle - \langle \beta \ , \ dL_{g^{-1}}(a_g) \rangle \\ &+ \ \langle \lambda \ , \ [dL_{g^{-1}}(a_g) \ , \ dL_{g^{-1}}(b_g)] \rangle \end{aligned} \tag{1.3}$$

Let us now restrict the situation on the subspace $\mu_1^{-1}(\lambda_0) \cong \{g, Ad^*_{g^{-1}}(\lambda_0))\} \subset G^{\mathbb{C}} \times (\mathfrak{g}^{\mathbb{C}})^*$. The tangent space $T_{(g,\lambda_0)}(G^{\mathbb{C}} \times \lambda_0)$ is isomorphic to the tangent space $T_g G^{\mathbb{C}}$. Each $a_g \in T_g G^{\mathbb{C}}$ can be expressed as $dR_g a$ for some $a \in \mathfrak{g}^{\mathbb{C}}$. Let $\imath : \mu_1^{-1}(\lambda_0) \to T^*G^{\mathbb{C}}$ be the natural inclusion and let $\imath^*(\omega)$ be the pull-back of the symplectic form ω. From the equation 1.3 we then get

$$\begin{aligned} \imath^*(\omega)_{(g,\lambda_0)}(\tilde{X}, \tilde{Y}) &= \omega_{(g,\lambda_0)}(d\imath(\tilde{X}), d\imath(\tilde{Y})) \\ &= \omega_{(g,\lambda_0)}(a_g, b_g) \\ &= \langle \lambda_0 \ , \ [dL_{g^{-1}} dR_g(a) \ , \ dL_{g^{-1}} dR_g(b)] \rangle \\ &= \langle \lambda_0 \ , \ [Ad_{g^{-1}}(a) \ , \ Ad_{g^{-1}}(b)] \rangle \\ &= \langle Ad^*_{g^{-1}}(\lambda_0) \ , \ [\ a \ , \ b \ ] \rangle \end{aligned} \tag{1.4}$$

We can now finally write the expression for the induced symplectic form ω_{KK} on the coadjoint orbit $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$. The projection $pr : \mu_1^{-1}(\lambda_0) \to \mathcal{O}^{\mathbb{C}}_{\lambda_0}$ is given by the formula

$$pr(g, \lambda_0) = Ad^*_{g^{-1}}(\lambda_0) \, .$$

The form ω_{KK} has to satisfy the condition $\imath^*(\omega) = pr^*(\omega_{KK})$. The derivative of the map pr is given by the formula $dpr_{(g,\lambda_0)}(Ad_{g^{-1}}\gamma) = -\{Ad_{g^{-1}}\gamma \ , \ Ad^*_{g^{-1}}(\lambda_0)\}$, where $\{\ \cdot \ , \ \cdot \ \}$ denotes the coadjoint action of $\mathfrak{g}^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}*}$ via the real pairing of $\mathfrak{g}^{\mathbb{C}}$ and $(\mathfrak{g}^{\mathbb{C}})^*$. In order to reduce the amount of symbols used, we are going to write the formula for ω_{KK} at the point $\lambda_0 \in \mathcal{O}^{\mathbb{C}}_{\lambda_0}$. Clearly, this will determine ω_{KK} at every other point of the orbit and the expression will be essentially the same. Since $pr(e, \lambda_0) = \lambda_0$, we get from 1.4 and from $pr^*(\omega_{KK}) = \imath^*(\omega)$

$$\begin{aligned} (pr^*\omega_{KK})_{(e,\lambda_0)}(a, b) &= (\omega_{KK})_{\lambda_0}(dpr(a) \ , \ dpr(b)) \\ &= (\omega_{KK})_{\lambda_0}(\{a, \lambda_0\} \ , \ \{b, \lambda_0\}) \\ &= \langle \lambda_0 \ , [\ a \ , \ b \ ] \rangle \end{aligned} \tag{1.5}$$

Now, every element α in the tangent space $T_{\lambda_0}\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ is of the form $\{a, \lambda_0\}$ for some $a \in \mathfrak{g}^{\mathbb{C}}$, so we can finally write the formula for ω_{KK}

$$(\omega_{KK})_{\lambda_0}(\alpha, \beta) = \langle \lambda_0 \ , \ [\ a \ , \ b\ ] \rangle \ , \tag{1.6}$$

where the equalities $\alpha = \{a, \lambda_0\}$ and $\beta = \{b, \lambda_0\}$ hold. (The elements a and b are of course not uniquely determined.) The form ω_{KK} is the well-known Kostant-Kirillov symplectic form on $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$.

(ii) The canonical form of the cotangent bundle

Let now B be a Borel subgroup of $G^{\mathbb{C}}$ and let B act on $G^{\mathbb{C}}$ by left translations. The lifting of this action on $T^*G^{\mathbb{C}}$ is again symplectic, and the appropriate moment map is given by a similar formula as before

$$\mu_2(g, \lambda) = \imath^*\lambda.$$

Here $\imath^* : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{b}$ is the projection dual to the natural inclusion $\imath : \mathfrak{b} \to \mathfrak{g}^{\mathbb{C}}$. Denote by $\mathfrak{n} = \mathfrak{b}^0 \subset (\mathfrak{g}^{\mathbb{C}})^*$ the polar of the Borel sub-algebra $\mathfrak{b} \in \mathfrak{g}^{\mathbb{C}}$. Then we obviously have $\mu_2^{-1}(0) = \{(g, \lambda); \lambda \in \mathfrak{n}\}$. Since the centraliser of $0 \in \mathfrak{b}^*$ is the whole group B, the symplectic quotient is simply

$$\mu_2^{-1}(0)/B = T^*(G^{\mathbb{C}}/B).$$

The induced symplectic form on $T^*(G^{\mathbb{C}}/B)$ is just the canonical cotangent symplectic form ω_{can}. Since $\omega_{can} = d\alpha$, where α again denotes the tautological 1-form, we have

$$[\omega_{can}] = [0] \in H^2(T^*(G^{\mathbb{C}}/B)).$$

As we shall show later in the text, the manifolds $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ and $T^*(G^{\mathbb{C}}/B)$ are diffeomorphic.

Proposition 2 *Let $G^{\mathbb{C}}/H$ be the quotient of a complex semi-simple Lie group and a Cartan subgroup. Let $\mathfrak{g}^{\mathbb{C}} = Lie(G^{\mathbb{C}})$. Then the cohomology group $H^2((G^{\mathbb{C}}/H);\mathbb{R})$ can be identified with the Cartan sub-algebra $\mathfrak{t} \subset \mathfrak{g}^{\mathbb{C}}$.*

Proof: The space $G^{\mathbb{C}}/H$ is the base space of the fibration $G^{\mathbb{C}} \longrightarrow G^{\mathbb{C}}/H$ with the fibre H. From the part of the basic exact sequence of the spectral sequence

$$\ldots \to H^1(G^{\mathbb{C}}/H) \to H^1(G^{\mathbb{C}}) \to H^1(H) \xrightarrow{\tau} H^2(G^{\mathbb{C}}/H) \to H^2(G^{\mathbb{C}}) \to \ldots,$$

and from the fact $H^1(G^{\mathbb{C}}) = H^2(G^{\mathbb{C}}) = 0$ which is a consequence of the semi-simplicity of $G^{\mathbb{C}}$ we conclude that

$$\tau : H^1(H) \longrightarrow H^2(G^{\mathbb{C}}/H)$$

is an isomorphism. Let $T \subset H$ be the maximal compact subgroup, i.e. a maximal torus in the compact real form $G \subset G^{\mathbb{C}}$. Then we have $H^1(H) \cong H^1(T)$ and clearly, we can identify $H^1(T)$ with $\mathfrak{t}^*$. The isomorphism τ is the usual transgression homomorphism. So $\tau(\lambda) = \beta$ means that λ is a restriction on $\mathfrak{h}$ of the form $\widetilde{\lambda} \in (\mathfrak{g}^{\mathbb{C}})^*$, such that

$$d\widetilde{\lambda} = \pi^*\beta,$$

where $\pi : G^{\mathbb{C}} \longrightarrow G^{\mathbb{C}}/H$ is the projection.

Assume now that we realize the homogeneous space $G^{\mathbb{C}}/H$ as the coadjoint orbit $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ of the regular semi-simple element $\lambda_0 \in (\mathfrak{g}^{\mathbb{C}})^*$. Comparing the construction of the Kostant-Kirillov form ω_{KK} with the transgression isomorphism we observe, that

$$\tau(\lambda_0) = \omega_{KK}.$$

We can realize the space $G^{\mathbb{C}}/H$ as the coadjoint orbit of any regular element lying in the chosen dual Cartan sub-algebra $\mathfrak{t}^* \subset (\mathfrak{g}^{\mathbb{C}})^*$, that is, of any element in $\mathfrak{t}^*$ away from the walls of the Weyl chambers in $\mathfrak{t}^*$. Since we have seen how to identify $H^2(G^{\mathbb{C}}/H)$ with $\mathfrak{t}^*$ we have proved, that a generic class in $H^2(G^{\mathbb{C}}/H)$ is indeed represented by a symplectic form on $G^{\mathbb{C}}/H$. □

1.2.2

The point $0 \in \mathfrak{t}$ is not regular. In fact it is the most exceptional one in the sense that it has the largest stabiliser with respect to the coadjoint action of $G^{\mathbb{C}}$. Clearly the whole $G^{\mathbb{C}}$ is the stabiliser. Nevertheless, the zero class $[0] \in H^2(G^{\mathbb{C}}/H)$ is also represented by a symplectic structure on $G^{\mathbb{C}}/H = \mathcal{O}^{\mathbb{C}}_{\lambda_0}$. As we shall see the comparison between the symplectic structure corresponding to $[0] \in H^2(G^{\mathbb{C}}/H)$ and a generic one is not canonical. It depends on the choice of a connection A on the principal bundle $\pi : G^{\mathbb{C}} \to G^{\mathbb{C}}/B$. Denote by A_{λ_0} the 1-form $\lambda_0 \circ A$ on $G^{\mathbb{C}}$, and let the 2-form γ_{λ_0} satisfy the condition

$$dA_{\lambda_0} = \pi^*(\gamma_{\lambda_0}) \ .$$

Finally, let β_{λ_0} be defined by $\beta_{\lambda_0} = pr^*(\gamma_{\lambda_0})$, where $pr : T^*(G^{\mathbb{C}}/B) \to G^{\mathbb{C}}/B$. The rest of this section is devoted to the proof of the following proposition.

Proposition 3 *The coadjoint orbit $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ of a regular element $\lambda_0 \in (\mathfrak{g}^{\mathbb{C}})^*$ and the cotangent bundle $T^*(G^{\mathbb{C}}/B)$ are diffeomorphic spaces. Moreover, there exists a 2-form β_{λ_0} on $T^*(G^{\mathbb{C}}/B)$ such that*

$$P : (\mathcal{O}^{\mathbb{C}}_{\lambda_0} \ , \ \omega_{KK}) \longrightarrow (T^*(G^{\mathbb{C}}/B) \ , \ \omega_{can} + \beta_{\lambda_0})$$

is a symplectic diffeomorphism for some appropriate map P depending on the choice of connection A on $\pi : G^{\mathbb{C}} \to G^{\mathbb{C}}/B$. Here $\beta_{\lambda_0} = pr^*(\gamma_{\lambda_0})$, and

$$(\gamma_{\lambda_0})_{[e]}(a, b) = b \cdot A_{\lambda_0}(a) - a \cdot A_{\lambda_0}(b)$$

where $[e] = \pi(e)$ *and* $a, b \in T_{[e]}G^{\mathbb{C}}/B$.

The strategy of the proof is based on [D-E-T] We will brake the argument into three steps. First we are going to construct another symplectic quotient M_{λ_0} of $T^*G^{\mathbb{C}}$. We will than compare M_{λ_0} to $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ on the one hand, and to $T^*(G^{\mathbb{C}}/H_{\lambda_0})$ on the other.

Step 1: Construction of M_{λ_0}

Let as before $\mathfrak{h}_{\lambda_0}$ be the Cartan sub-algebra corresponding to H_{λ_0}. Let Δ be the system of roots in $\mathfrak{h}_{\lambda_0}$ and $\Delta = \Delta_+ + \Delta_-$ some decomposition of Δ on positive and negative roots. Denote by $\mathfrak{b}_{\lambda_0}$ the Borel sub-algebra

$$\mathfrak{b}_{\lambda_0} = \mathfrak{h}_{\lambda_0} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

Lemma 2 *Let* $i : \mathfrak{b}_{\lambda_0} \hookrightarrow \mathfrak{g}^{\mathbb{C}}$ *be the natural inclusion and* $i^* : \mathfrak{g}^{\mathbb{C}^*} \hookrightarrow \mathfrak{b}^*_{\lambda_0}$ *the projection adjoint to* i*. Then* $i^*(\lambda_0) = \nu_0$ *is invariant with respect to the coadjoint action of* B_{λ_0}.

Proof of lemma 2: Since $B_{\lambda_0} = exp(\mathfrak{b}_{\lambda_0})$ it suffices to show that ν_0 is invariant with respect to the coadjoint action of the Lie algebra $\mathfrak{b}_{\lambda_0}$.

Since $[\mathfrak{b}_{\lambda_0}, \mathfrak{b}_{\lambda_0}] = \mathfrak{n}_{\lambda_0} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, we clearly have

$$\langle \nu_0, [\mathfrak{b}_{\lambda_0}, \mathfrak{b}_{\lambda_0}] \rangle = 0.$$

Denote the coadjoint action of x on ν_0 by $\{x, \nu_0\}$. Fixing an $x \in \mathfrak{b}_{\lambda_0}$, we have for every $y \in \mathfrak{b}_{\lambda_0}$

$$\langle \nu_0, [x, y] \rangle = \langle \{\nu_0, x\}, y \rangle = 0$$

and therefore $\{\nu_0, x\} = 0$.

□

Let the group B_{λ_0} act from the left on $T^*G^{\mathbb{C}}$ and let $\mu_B : T^*G^{\mathbb{C}} \longrightarrow \mathfrak{b}^*_{\lambda_0}$ be the moment map of this action. The formula for this map is

$$\mu_B(g, \lambda) = i^*(\lambda).$$

(Again we trivialised $T^*G^{\mathbb{C}}$ by the left translations) The above lemma tells us that the stabiliser of $\nu_0 \in \mathfrak{b}^*_{\lambda_0}$ with respect to the coadjoint action of B_{λ_0} is the whole group B_{λ_0}, and this allows us to form the symplectic quotient

$$\mu_B^{-1}(\nu_0)/B_{\lambda_0} = M_{\lambda_0}.$$

We will denote the induced symplectic form on M_{λ_0} by ω_{λ_0}.

For later use we have to describe the preimage $\mu_B^{-1}(\lambda_0)$ under the trivialisation by the left translations. Let (g, λ) be an element in the preimage and let $\alpha \in \mathfrak{b}_{\lambda_0}$ be arbitrary. Writing $\lambda = \lambda_0 + \gamma$ we get

$$\begin{aligned} \langle \lambda_0, \alpha \rangle &= \langle \imath^*(\lambda), \alpha \rangle \\ &= \langle \lambda_0 + \gamma, \imath(\alpha) \rangle \\ &= \langle \lambda_0, \alpha \rangle + \langle \gamma, \alpha \rangle \end{aligned}$$

Therefore $\langle \gamma, \alpha \rangle = 0$ for every $\alpha \in \mathfrak{b}_{\lambda_0}$. From this we see that $\mu_b^{-1}(\lambda_0) = G^{\mathbb{C}} \times (\lambda_0 + \mathfrak{n}^{\lambda_0}) \cong G^{\mathbb{C}} \times \mathfrak{n}^{\lambda_0}$.

Step 2: Symplectomorphism between M_{λ_0} and $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$

Lemma 3 *There exists a symplectic action of $G^{\mathbb{C}}$ on M_{λ_0}.*

Proof of lemma 3: Let $G^{\mathbb{C}}$ act on itself from the left. Since B_{λ_0} acted from the right, the two actions commute, and so do their liftings on $T^*G^{\mathbb{C}}$. The moment map $\mu_{G^{\mathbb{C}}} : T^*G^{\mathbb{C}} \longrightarrow (\mathfrak{g}^{\mathbb{C}})^*$ of our new action is given by the formula

$$\mu_{G^{\mathbb{C}}}(g, \lambda) = Ad_g^* \lambda$$

Clearly, the $G^{\mathbb{C}}$-action leaves invariant the space $\mu_B^{-1}(\nu_0)$ (in the right trivialisation this action is given by $h \cdot (g, \lambda) = (gh, \lambda)$) and it therefore descends on a symplectic $G^{\mathbb{C}}$-action on the symplectic quotient M_{λ_0}. The action is symplectic since it is induced by a symplectic action on $T^*G^{\mathbb{C}}$ on the symplectic quotient M_{λ_0}.

$\square$

Lemma 4 *The orbit $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ contains the affine plane $\lambda_0 + \mathfrak{n}^{\lambda_0}$, where $\mathfrak{n}^{\lambda_0} \subset (\mathfrak{g}^{\mathbb{C}})^*$ is the annihilator of the Borel sub-algebra $\mathfrak{b}_{\lambda_0} \subset \mathfrak{g}^{\mathbb{C}}$ with respect to the natural pairing. More precisely, $\lambda_0 + \mathfrak{n}^{\lambda_0} = Ad^*_{B_{\lambda_0}}(\lambda_0)$, where B_{λ_0} is the Borel subgroup corresponding to the sub-algebra $\mathfrak{b}_{\lambda_0}$.*

Proof of lemma 4: We will show, that

$$\lambda_0 + \mathfrak{n}^{\lambda_0} = Ab^*_{B_{\lambda_0}}(\lambda_0).$$

Lemma then follows automatically.

Choose an element $b \in B_{\lambda_0}$ and write $Ad^*_b(\lambda_0) = \lambda_0 + \gamma$. Take an arbitrary element $\alpha \in \mathfrak{b}_{\lambda_0}$ and consider the pairing

$$\begin{aligned} \langle Ad^*_b(\lambda_0), \alpha\rangle &= \langle \lambda_0 + \gamma, \alpha\rangle \\ &= \langle \lambda_0, \alpha\rangle + \langle \gamma, \alpha\rangle \qquad (1.7) \\ &= \langle \lambda_0, Ad_b(\alpha)\rangle \end{aligned}$$

Define the map $f : B_{\lambda_0} \to (\mathfrak{g}^{\mathbb{C}})^*$ by the formula $f(b) = \langle \lambda_0, Ad_b(\alpha)\rangle$ for a fixed $\alpha \in \mathfrak{g}^{\mathbb{C}}$. At an arbitrary point $b_0 \in B_{\lambda_0}$ the derivative of f is given by

$$df_{b_0}(X) = \langle \lambda_0, [X, Ad_{b_0}(\alpha)]\rangle = 0\,,$$

where $X \in T_{b_0}B_{\lambda_0} \cong \mathfrak{b}_{\lambda_0}$. Since its derivative is everywhere zero, f is a constant mapping, and therefore $\langle \lambda_0, Ad_b(\alpha)\rangle = \langle \lambda_0, \alpha\rangle$. From 1.7 we then get $\langle \gamma, \alpha\rangle = 0$. This being true for every $\alpha \in \mathfrak{b}_{\lambda_0}$, the element γ indeed lies in the annihilator $\mathfrak{n}^{\lambda_0}$ of $\mathfrak{b}_{\lambda_0}$, which proves the inclusion $Ad^*_{B_{\lambda_0}}(\lambda_0) \subset \lambda_0 + \mathfrak{n}^{\lambda_0}$.

In the n-dimensional Lie group of rank r the dimension of a Borel subgroup is $(1/2)(n+r)$ and that of its annihilator $(1/2)(n-r)$. The stabiliser of λ_0 in B_{λ_0} is the Cartan subgroup H_{λ_0} which is r-dimensional, so the dimension of the homogeneous space $Ad_{B_{\lambda_0}}(\lambda_0)$ is $(1/2)(n-r) = dim(\lambda_0 + \mathfrak{n}^{\lambda_0})$. So $Ad_{B_{\lambda_0}}(\lambda_0)$ is an open subset in $\lambda_0 + \mathfrak{n}^{\lambda_0}$. Since it is also a closed subset, we finally have $Ad_{B_{\lambda_0}}(\lambda_0) = \lambda_0 + \mathfrak{n}^{\lambda_0}$.

One could also prove the inclusion $\lambda_0 + \mathfrak{n}^{\lambda_0} \subset \mathcal{O}^{\mathbb{C}}_{\lambda_0}$ using the invariant polynomials. Let $(\mathfrak{g}^{\mathbb{C}})^*$ be identified with $\mathfrak{g}^{\mathbb{C}}$ via the Killing form $\mathcal{K}$. Denote the image of $\lambda_0 + \mathfrak{n}^{\lambda_0}$ with respect to this identification by $\#\lambda_0 + \mathfrak{n}^{\lambda_0}$. Represent $\mathfrak{g}$ by the ad-representation. In a right choice of the basis in $\mathfrak{g}^{\mathbb{C}}$ (the one dual to a simple system of roots in Δ) the elements of $\#(\lambda_0 + \mathfrak{n}^{\lambda_0}$ will be upper-triangular matrices with the representative $\widetilde{\lambda_0}$ of λ_0 on the diagonal. The orbit $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ corresponds to the elements $x \in \mathfrak{g}^{\mathbb{C}}$ such that $p_i(x) = p_i(\lambda_0)$, $i = 1, \ldots, r$, where $\{p_i\}_{i=1}^r$ is some basis of invariant functions on $\mathfrak{g}^{\mathbb{C}}$. Since in an upper-triangular element x the values $p(x)$ depend only on the terms on the diagonal, we have $p_i(x) = p(\widetilde{\lambda_0})$ for every i and every element $x \in \#(\lambda_0 + \mathfrak{n}^{\lambda_0})$, which proves $\lambda_0 + \mathfrak{n}^{\lambda_0} \subset \mathcal{O}^{\mathbb{C}}_{\lambda_0}$.

□

The above result will be used in the proof of the following two lemmas.

Lemma 5 *The moment map*

$$\mu_{G^{\mathbb{C}}} : M_{\lambda_0} \longrightarrow \mathcal{O}^{\mathbb{C}}_{\lambda_0}$$

is a diffeomorphism.

Proof of lemma 5: First we see directly that

$$Im(\mu_{G^{\mathbb{C}}}) = \{Ad^*_g(\lambda); g \in G^{\mathbb{C}}, \lambda \in \lambda_0 + (n)^{\lambda_0}\},$$

and so we clearly have $\mathcal{O}^{\mathbb{C}}_{\lambda_0} \subset Im(\mu_{G^{\mathbb{C}}})$. On the other hand, the $G^{\mathbb{C}}$-action on M_{λ_0} is obviously transitive. The equivariance of our moment map then implies $Im(\mu_{G^{\mathbb{C}}}) = \mathcal{O}^{\mathbb{C}}_{\lambda_0}$. The equality

$$Ad_{B_{\lambda_0}}(\lambda_0) = \lambda_0 + \mathfrak{n}_{\lambda_0} \tag{1.8}$$

proved above will enable us to prove the injectivity of the map $\mu_{G^{\mathbb{C}}}$. Suppose we have (g_1, λ_1) and (g_2, λ_2) in $\mu_B^{-1}(\nu_0) = G^{\mathbb{C}} \times (\lambda_0 + \mathfrak{n}_{\lambda_0})$ such that $\mu_{G^{\mathbb{C}}}(g_1, \lambda_1) = \mu_{G^{\mathbb{C}}}(g_2, \lambda_2)$. This would mean that $Ad^*_{g_1}(\lambda_1) = Ad^*_{g_2}(\lambda_2)$. The equality 1.8 gives us the existence of elements $b_1, b_2 \in B_{\lambda_0}$ such that $\lambda_1 = Ad^*_{b_1}(\lambda_0)$ and $\lambda_2 = Ad^*_{b_2}(\lambda_0)$. So we have $b_1^{-1}g_1^{-1}g_2b_2 \in H_{\lambda_0}$. Since $H_{\lambda_0} \subset B_{\lambda_0}$ this means that $b_0 = g_1^{-1}g_2$ is an element of B_{λ_0}. The elements (g_1, λ_1) and (g_2, λ_2) therefore lie in the same B_{λ_0}-orbit, (they differ by b_0), which proves the injectivity of the map $\mu_{G^{\mathbb{C}}}$.

□

Lemma 6 *The moment map*

$$\mu_{G^{\mathbb{C}}} : (M_{\lambda_0}, \omega_{\lambda_0}) \longrightarrow (\mathcal{O}^{\mathbb{C}}_{\lambda_0}, \omega_{KK})$$

is a symplectomorphism, i.e. $\mu^*_{G^{\mathbb{C}}}(\omega_{KK}) = \omega_{\lambda_0}$.

Proof of lemma 6: First we shall describe the form ω_{λ_0}, and then compare it to the pull-back of ω_{KK}. Let again $T^*G^{\mathbb{C}}$ be trivialised by the left translations, giving the isomorphisms $T^*G^{\mathbb{C}} \cong G^{\mathbb{C}} \times (\mathfrak{g}^{\mathbb{C}})^*$, and $\mu_B^{-1}(\lambda_0) \cong G^{\mathbb{C}} \times (\lambda_0 + \mathfrak{n}^{\lambda_0})$. We will work at a point of the form $(e, \lambda_0 + \gamma) \in G^{\mathbb{C}} \times (\lambda_0 + \mathfrak{n}^{\lambda_0})$. The proof of the lemma at a general point $(g, \lambda_0 + \gamma)$ will then follow immediately.

Let $\imath : \mu_B^{-1}(\lambda_0) \to T^*G^{\mathbb{C}}$ be the natural inclusion. From the formula 1.3 we get the following expression for the restriction $\imath^*(\omega)$ of the symplectic form ω.

$$\begin{aligned} \imath^*(\omega)_{(e,\lambda_0+\gamma)}((a,\alpha),(b,\beta)) &= \langle \alpha, b\rangle - \langle \beta, a\rangle \\ &+ \langle (\lambda_0+\gamma),[a,b]\rangle \end{aligned} \tag{1.9}$$

Here (a,α) and (b,β) are elements of the tangent space $T_{(e,\lambda_0+\gamma)}(G^{\mathbb{C}} \times (\lambda_0 + \mathfrak{n}^{\lambda_0}))$ which is isomorphic to $\mathfrak{g}^{\mathbb{C}} \times \mathfrak{n}^{\lambda_0}$.

Let now ω_{KK} be Kostant-Kirillov form on $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ and $\mu^*_{G^{\mathbb{C}}}(\omega_{KK})$ its pull-back. Then we have

$$\mu^*_{G^{\mathbb{C}}}(\omega_{KK})_{(e,\lambda_0+\gamma)}((a,\alpha),(b,\beta)) = (\omega_{KK})_{\lambda_0}(d\mu_{G^{\mathbb{C}}}(a,\alpha), d\mu_{G^{\mathbb{C}}}(b,\beta)) \tag{1.10}$$

Let $t \to (g(t), \lambda_0+\delta(t))$ be a path in $G^{\mathbb{C}} \times \mathfrak{n}^{\lambda_0}$, such that $g(0)=e$, $\lambda_0+\delta(0)=\lambda_0+\gamma$, and $\frac{d}{dt}|_{t=0}g(t) = a$, $\frac{d}{dt}|_{t=0}(\lambda_0+\delta(t)) = \alpha$. The derivative of the map $\mu_{G^{\mathbb{C}}}$ is then given by the formula

$$\begin{aligned} (d\mu_{G^{\mathbb{C}}})_{(e,\lambda_0+\gamma)}(a,\alpha) &= \tfrac{d}{dt}|_{t=0} Ad^*_{g(t)}(\lambda_0+\delta(t)) \\ &= \{a, \lambda_0+\gamma\} + \alpha \end{aligned} \tag{1.11}$$

The elements α and β lie in the tangent space $T_{(\lambda_0+\gamma)}\mathcal{O}^{\mathbb{C}}_{\lambda_0}$, so there exist $\widetilde{\alpha}, \widetilde{\beta} \in \mathfrak{g}^{\mathbb{C}}$ such that $ad^*_{\lambda_0+\gamma}(\widetilde{\alpha}) = \alpha$ and $ad^*_{\lambda_0+\gamma}(\widetilde{\beta}) = \beta$. Now we can put the formula 1.6 for Kostant-Kirillov form into 1.11 and get

$$\begin{aligned} \mu^*_{G^{\mathbb{C}}}(\omega_{KK})_{(e,\lambda_0+\gamma)}((a,\alpha),(b,\beta)) &= \langle \lambda_0+\gamma \ , \ [a+\widetilde{\alpha}, b+\widetilde{\beta}]\rangle \\ &= \langle \{a+\widetilde{\alpha}, \lambda_0+\gamma\} \ , \ b+\widetilde{\beta}\rangle \\ &= \langle \{a, \lambda_0+\gamma\} \ , \ b\rangle + \langle \{a, \lambda_0+\gamma\} \ , \ \widetilde{\beta}\rangle \\ &+ \langle \{\widetilde{\alpha}, \lambda_0+\gamma\} \ , \ b\rangle + \langle \{\widetilde{\alpha}, \lambda_0+\gamma\} \ , \ \widetilde{\beta}\rangle. \end{aligned}$$

The elements α and $\lambda_0+\gamma$ both lie in the annihilator $\mathfrak{n}^{\lambda_0}$ of $\mathfrak{b}_{\lambda_0}$, and since $\{\widetilde{\alpha}, \lambda_0+\gamma\} = \alpha$, the element $\widetilde{\alpha}$ lies in the sub-algebra $\mathfrak{b}_{\lambda_0}$. The same is true for $\widetilde{\beta}$. So the term $\langle \{\widetilde{\alpha}, \lambda_0+\gamma\} \ , \ \widetilde{\beta}\rangle = \langle \lambda_0+\gamma \ , \ [\widetilde{\alpha}, \widetilde{\beta}]\rangle$ is equal to zero. This finally gives us

$$\begin{aligned} \mu^*_{G^{\mathbb{C}}}(\omega_{KK})_{(e,\lambda_0+\gamma)}((a,\alpha),(b,\beta)) &= \langle \alpha \ , \ b\rangle - \langle \beta \ , \ a\rangle \\ &= \langle \lambda_0+\gamma \ , \ [a,b]\rangle \end{aligned} \tag{1.12}$$

Comparing the expressions 1.9 and 1.12 we see that the forms $\imath^*(\omega)$ and $\mu^*_{G^{\mathbb{C}}}(\omega_{KK})$ on $\mu_B^{-1}(\lambda_0)$ are the same, so they will descend onto the same form on the quotient $\mu_B^{-1}(\lambda_0)/B_{\lambda_0} = \mathcal{O}^{\mathbb{C}}_{\lambda_0}$ which proves the lemma.

□

Step 3: Comparison between M_{λ_0} and $T^*(G^{\mathbb{C}}/B_{\lambda_0})$

The second half of our task is to construct a diffeomorphism

$$R : M_{\lambda_0} \longrightarrow T^*(G^{\mathbb{C}}/B_{\lambda_0})$$

and a 2-form β_{λ_0} on $T^*(G^{\mathbb{C}}/B_{\lambda_0})$ such that R will be a symplectomorphism between $(M_{\lambda_0}, \omega_{\lambda_0})$ and $(T^*(G^{\mathbb{C}}/B_{\lambda_0}), \omega_{can} + \beta_{\lambda_0})$. As we shall see, the 2-form β_{λ_0} descends on the base space of the cotangent bundle, i.e. it is of the form $\beta_{\lambda_0} = pr^*(\gamma)$ for some 2-form γ on the space $G^{\mathbb{C}}/B_{\lambda_0}$. Here $pr : T^*(G^{\mathbb{C}}/B_{\lambda_0}) \longrightarrow G^{\mathbb{C}}/B_{\lambda_0}$ is the natural projection. First choose a connection A on the principal bundle

$$\begin{array}{ccc} B_{\lambda_0} \rightarrow & G^{\mathbb{C}} \\ & \downarrow \\ & G^{\mathbb{C}}/B_{\lambda_0} \end{array}$$

This means that A is a 1-form on $G^{\mathbb{C}}$ with values in $\mathfrak{b}_{\lambda_0}$, satisfying the usual equivariance condition and the identity on the vertical spaces. Trivialising the bundle $T^*G^{\mathbb{C}}$ by the right translations the connection A gives us a family of projections

$$A_g : \mathfrak{g}^{\mathbb{C}} \longrightarrow \mathfrak{b}_{\lambda_0}; \quad g \in G^{\mathbb{C}},$$

and dually a family of inclusions

$$A_g^* : \mathfrak{b}^*_{\lambda_0} \longrightarrow \mathfrak{g}^{\mathbb{C}*}; \quad g \in G^{\mathbb{C}}.$$

Denote as before $\nu_0 = i^*(\lambda_0)$. Since A_g is the identity on $\mathfrak{b}_{\lambda_0}$, we have for an arbitrary element $b \in \mathfrak{b}_{\lambda_0}$

$$\begin{aligned} \langle A_g^*(\nu_0)\ ,\ b\rangle &= \langle \lambda_0\ ,\ \ A_g(b)\rangle \\ &= \langle \lambda_0\ ,\ b\rangle \end{aligned}$$

This implies that for all $g \in G^{\mathbb{C}}$ we have $A_g(\nu_0) = \lambda_0 + n_g$ for some $n_g \in \mathfrak{n}^{\lambda_0}$, which allows us to define the mapping

$$\widetilde{R} : \mu_B^{-1}(\nu_0) = G^{\mathbb{C}} \times (\lambda_0 + \mathfrak{n}^{\lambda_0}) \longrightarrow G^{\mathbb{C}} \times \mathfrak{n}^{\lambda_0}$$

by the formula

$$\widetilde{R}(g,\lambda) = (g, \lambda - A^*_g(\nu_o)).$$

This map is obviously equivariant with respect to the action of B_{λ_0}, and it is easily seen that the induced map

$$R : M_{\lambda_0} \longrightarrow T^*(G^{\mathbb{C}}/B_{\lambda_0})$$

is a diffeomorphism. Denote by A_ν the composition $\nu_0 \circ A : TG^{\mathbb{C}} \to \mathbb{C}$. Since A is a connection and ν_0 is invariant with respect to the coadjoint action of B_{λ_0} on $\mathfrak{b}_{\lambda_0}$, the 1-form A_ν is invariant with respect to the natural action of B_{λ_0} on $G^{\mathbb{C}}$ from the right. So there exists a 2-form γ on $G^{\mathbb{C}}/B_{\lambda_0}$ such that

$$dA_\nu = \pi^*\gamma,$$

where $\pi : G^{\mathbb{C}} \to G^{\mathbb{C}}/B_{\lambda_0}$ is the projection. Denote $\beta_{\lambda_0} = pr^*(\gamma)$.

Lemma 7 *The mapping*

$$R : (M_{\lambda_0}, \omega_{\lambda_0}) \longrightarrow (T^*(G^{\mathbb{C}}/B_{\lambda_0}), \omega_{can} + \beta_{\lambda_0})$$

is a symplectomorphism. Here β_{λ_0} is of the form $pr^*(\gamma)$ and γ is given by the formula

$$\gamma_{[e]}(a,b) = b \cdot A_\nu(a) - a \cdot A_\nu(b)\ ,$$

where $[e] = \pi(e)$.

Proof of lemma 7: We have to prove that

$$R^*(\omega_{can} + \beta_{\lambda_0}) = \omega_{\lambda_0}.$$

We will show that the map $\widetilde{R}$ pulls back the form $\omega_{can} + \beta_{\lambda_0}$ lifted on $G^{\mathbb{C}} \times \mathfrak{n}^{\lambda_0}$ to the lifting of ω_{λ_0} on $G^{\mathbb{C}} \times (\lambda_0 + \mathfrak{n}^{\lambda_0})$. The lemma will then follow by the equivariance with respect to the action of the group B_{λ_0}.

Let (a, α) be a vector in $T_{(e,\lambda_0+\gamma)}(G^{\mathbb{C}} \times (\lambda_0 + \mathfrak{n}^{\lambda_0}))$, and let $t \to (g(t), \lambda(t))$ be a path in $G^{\mathbb{C}} \times (\lambda_0 + \mathfrak{n}^{\lambda_0})$, such that (a, α) is its tangent at the point $(e, \lambda_0 + \gamma)$. The derivative of the map $\widetilde{R}$ is then

$$\begin{aligned} d\widetilde{R}(a,\alpha) &= \tfrac{d}{dt}|_{t=0}\widetilde{R}(g(t),\lambda(t)) \\ &= \tfrac{d}{dt}|_{t=0}(g(t), \lambda(t) - A^*_{g(t)}(\nu_0)) \\ &= (a\ ,\ \alpha - \tfrac{d}{dt}|_{t=0}A^*_{g(t)}(\nu_0)) \end{aligned} \qquad (1.13)$$

Choose an arbitrary element $x \in \mathfrak{g}^{\mathbb{C}}$ and observe the function

$$f(t) = \langle A^*_{g(t)}(\nu_0), x\rangle = \langle \nu_0, A^*_{g(t)}(x)\rangle = (\nu_0 \circ A^*_{g(t)})(x)$$

defined by the path $t \to g(t)$. The derivative of this function at zero is given by

$$\frac{d}{dt}|_{t=0} f(t) = a \cdot A_\nu(x) ,$$

where $\cdot$ means the derivative of the function $(\nu_0 \circ A_g)(x) : G^{\mathbb{C}} \to \mathbb{C}$ at the point e in the direction a.

From the expressions 1.3 and 1.13 we get the following one.

$$\begin{aligned}
\widetilde{R}^*(\omega_{can})_{(e,\lambda)}((a,\alpha)\ ,\ (b,\beta)) &= \langle \alpha - \tfrac{d}{dt} A^*_{g(t)}(\nu_0)\ ,\ b\rangle - \langle \beta - \tfrac{d}{dt} A^*_{g(t)}(\nu_0)\ ,\ a\rangle \\
&\quad + \langle \lambda\ ,\ [a,b]\rangle \\
&= \langle \alpha\ ,\ b\rangle - \langle \beta\ ,\ a\rangle + \langle \lambda\ ,\ [a,b]\rangle \\
&\quad - \langle \tfrac{d}{dt} A^*_{g(t)}(\nu_0)\ ,\ b\rangle + \langle \tfrac{d}{dt} A^*_{g(t)}(\nu_0)\ ,\ a\rangle \\
&= \langle \alpha\ ,\ b\rangle - \langle \beta\ ,\ a\rangle + \langle \lambda\ ,\ [a,b]\rangle \\
&\quad - b \cdot A_\nu(a) + a \cdot A_\nu(b)
\end{aligned} \tag{1.14}$$

The form β_{λ_0} is lifted from γ via the projection

$$pr : T^*(G^{\mathbb{C}}/B_{\lambda_0}) \to G^{\mathbb{C}}/B_{\lambda_0} ,$$

so it will only act on the base components of the tangent space $T(T^*G^{\mathbb{C}}/B_{\lambda_0})$, i.e. it will act as γ. If we lift γ on $G^{\mathbb{C}}$ we get

$$\pi^*(\gamma)(a,b) = dA_\nu(a,b) = b \cdot A_\nu(a) - a \cdot A_\nu + A_\nu([a,b])$$

Now, since $A_\nu = \nu_0 \circ A_g$ and since A_g is a projection of $\mathfrak{g}^{\mathbb{C}}$ onto $\mathfrak{b}_{\lambda_0}$, we have $A_\nu([a,b]) = 0$. This together with the equation 1.14 and a slight abuse of notation gives us the desired equality

$$\widetilde{R}^*(\omega_{can} + \beta_{\lambda_0}) = \omega_{\lambda_0} .$$

□

Now we can finally define the symplectomorphism

$$P = R \circ \mu^{-1}_{G^{\mathbb{C}}} : (\mathcal{O}^{\mathbb{C}}_{\lambda_0}, \omega_{KK}) \longrightarrow T^*(G^{\mathbb{C}}/B_{\lambda_0}, \omega_{can} + \beta_{\lambda_0}),$$

which proves the Proposition 1 at the beginning of this subsection.

1.2.3

We shall conclude this section by generalising the proposition 3 to the case, where the initial symplectic space, whose quotients are compared, is the cotangent bundle T^*P, and

$$\begin{array}{ccc} G^{\mathbb{C}} & \longrightarrow & P \\ & & \downarrow \\ & & N \end{array}$$

is a principal $G^{\mathbb{C}}$-bundle. Suppose that $G^{\mathbb{C}}$ acts on P from the right. Then we can lift this action on the cotangent bundle T^*P to get a Hamiltonian action with the moment map

$$\widetilde{\mu_G} : P \longrightarrow (\mathfrak{g}^{\mathbb{C}})^*$$

Let λ_0 as before be a regular element in $(\mathfrak{g}^{\mathbb{C}})^*$ and also a regular value of the map $\widetilde{\mu_G}$ and let H_{λ_0} be the stabiliser of λ_0 with respect to the coadjoint action. Denote the symplectic quotient $\widetilde{\mu_G}^{-1}(\lambda_0)/H_{\lambda_0}$by $\mathcal{PO}^{\mathbb{C}}_{\lambda_0}$, and the induced symplectic form by $\omega_{\mathcal{P}KK}$.

Next, let $B_{\lambda_0} \subset G^{\mathbb{C}}$ be the Borel subgroup corresponding to the Borel sub-algebra $\mathfrak{b}_{\lambda_0} = \mathfrak{h}_{\lambda_0} \oplus \bigoplus_{\alpha \in R+} \mathfrak{g}_\alpha$. Then P is of course also a B_{λ_0}-principal bundle and we can form different symplectic quotients of T^*P with respect to the lifted B_{λ_0}-action. Let

$$\widetilde{\mu_B} : T^*P \longrightarrow (\mathfrak{b}_{\lambda_0})^*$$

be the moment map of the B_{λ_0}-action on T^*P. Then, as always in such cases, the symplectic quotient $\widetilde{\mu_B}^{-1}(0)/B_{\lambda_0}$ is symplectically isomorphic to the cotangent bundle $T^*(P/B_{\lambda_0})$ with the canonical symplectic form.

Proposition 4 *The symplectic quotients $\mathcal{PO}^{\mathbb{C}}_{\lambda_0}$ and $T^*(P/B_{\lambda_0})$ are diffeomorphic spaces. Moreover there exist a diffeomorphism R and a 2-form β_{λ_0} on $T^*(P/B_{\lambda_0})$ such that*

$$R : (\mathcal{PO}^{\mathbb{C}}_{\lambda_0} \ , \ \omega_{\mathcal{P}KK}) \longrightarrow (T^*(P/B_{\lambda_0}) \ , \ \omega_{can} + \beta_{\lambda_0})$$

is a symplectomorphism. The form β_{λ_0} is a pull-back $pr^*(\gamma_{\lambda_0})$ of a 2-form γ_{λ_0} on P by the natural projection $pr : T^*(P/B_{\lambda_0}) \to (P/B_{\lambda_0})$. The latter in turn satisfies the condition

$$dA_{\lambda_0} = \pi^*(\gamma_{\lambda_0}) \ ,$$

where the 1-form A_{λ_0} on P is defined by $A_{\lambda_0} = \lambda_0 \circ A$ for some connection A on the principal B_{λ_0}-bundle $\pi : P \to P/B_{\lambda_0}$.

Proof: The proof is essentially the same as the proof of proposition 3. We will compare the "intermediate" symplectic quotient $\widetilde{\mu_B}^{-1}(\lambda_0)/B_{\lambda_0}$ denoted by $\mathcal{P}M_{\lambda_0}$ to

$\mathcal{P}\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ on one hand and to $T^*(P/B_{\lambda_0})$ on the other. The existence of $\mathcal{P}M_{\lambda_0}$ follows directly from step 1 of the proof of proposition 3.

First we shall construct the symplectomorphism

$$F : (\mathcal{P}M_{\lambda_0}\ ,\ \omega_{\lambda_0}) \longrightarrow (\mathcal{P}\mathcal{O}^{\mathbb{C}}_{\lambda_0}\ ,\ \omega_{\mathcal{P}KK})\ . \tag{1.15}$$

Let the group $G^{\mathbb{C}}$ act on the product $P \times G^{\mathbb{C}}$ by the action $h \cdot (p, g) = (ph, h^{-1}g)$. Then the quotient space is a fibre bundle associated to the principal bundle P with the fibre $G^{\mathbb{C}}$ and with $G^{\mathbb{C}}$ acting from the right, i.e. the quotient space is again the principal bundle P. We can lift the above action on the cotangent bundle $T^*(P \times G^{\mathbb{C}})$. Let μ_P be the moment map of this action. Then the symplectic quotient $\mu_P^{-1}(0)/G^{\mathbb{C}}$ is the cotangent bundle $T^*((P \times G^{\mathbb{C}})/G^{\mathbb{C}}) = T^*P$ with the canonical symplectic form. Let on the other hand B_{λ_0} act from the right on the second factor of $T^*(P \times G^{\mathbb{C}}) = T^*P \times T^*G^{\mathbb{C}}$. The corresponding symplectic quotient $\mu_B^{-1}(\lambda_0)/B_{\lambda_0}$ is the space M_{λ_0}. The $G^{\mathbb{C}}$-action on $T^*(P \times G^{\mathbb{C}})$ descends on a Hamiltonian $G^{\mathbb{C}}$-action on the space $T^*P \times M_{\lambda_0}$. The corresponding moment map

$$\widetilde{\mu_1} : T^*P \times M_{\lambda_0} \longrightarrow (\mathfrak{g}^{\mathbb{C}})^*$$

is given by $\widetilde{\mu_1}(\lambda_p, x) = \widetilde{\mu_G}(\lambda_p) + \mu_{G^{\mathbb{C}}}(x)$. By $\mu_{G^{\mathbb{C}}}$ we denote again the map

$$\mu_{G^{\mathbb{C}}} : M_{\lambda_0} \longrightarrow \mathcal{O}^{\mathbb{C}}_{\lambda_0}$$

discussed in the second step in the proof of proposition 3. Since the image of $\mu_{G^{\mathbb{C}}}$ is $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$, we see that the elements (λ_p, x) in the preimage

$$\widetilde{\mu_1}^{-1}(0) \subset \widetilde{\mu_G}^{-1}(\mathcal{O}^{\mathbb{C}}_{\lambda_0}) \times M_{\lambda_0}\ . \tag{1.16}$$

satisfy the condition

$$\widetilde{\mu_G}(\lambda_p) = -\mu_{G^{\mathbb{C}}}(x)\ . \tag{1.17}$$

Recall that for an arbitrary symplectic space M equipped with a Hamiltonian G-action and the corresponding moment map $\mu : M \to \mathfrak{g}^*$ we have the equality

$$\mu^{-1}(\lambda_0)/H_{\lambda_0} = \mu^{-1}(\mathcal{O}_{\lambda_0})/G\ ,$$

where $\mathcal{O}_{\lambda_0}$ is the coadjoint orbit through λ_0. From this and from 1.16, and 1.17 it is then easily seen that

$$\widetilde{\mu_1}^{-1}(0)/G^{\mathbb{C}} = \mathcal{P}M_{\lambda_0}\ .$$

Mutatis mudandis we can repeat the above construction for the quotient $\mathcal{P}\mathcal{O}^{\mathbb{C}}_{\lambda_0}$. Let $\nu_G : T^*G^{\mathbb{C}} \to (\mathfrak{g}^{\mathbb{C}})^*$ be the moment map of the right $G^{\mathbb{C}}$-action on $T^*G^{\mathbb{C}}$ and

$\mathcal{O}^{\mathbb{C}}_{\lambda_0} = \nu_G^{-1}(\lambda_0)/H_{\lambda_0}$. Then we have the induced $G^{\mathbb{C}}$-action on $T^*G^{\mathbb{C}} \times \mathcal{O}^{\mathbb{C}}_{\lambda_0}$. This action is also Hamiltonian having

$$\widetilde{\mu_2} : T^*P \times \mathcal{O}^{\mathbb{C}}_{\lambda_0} \longrightarrow (\mathfrak{g}^{\mathbb{C}})^*$$

as the moment map. This map is given by $\widetilde{\mu_2}(\lambda_p, x) = \widetilde{\mu_G}(\lambda_p) + \mu_G(x)$. The symbol μ_G now denotes the moment map of the left lifted $G^{\mathbb{C}}$-action on $T^*G^{\mathbb{C}}$. Restriction of μ_G on $\mathcal{O}^{\mathbb{C}}_{\lambda_0}$ is just the identity map. This time we get that the elements (λ_p, x) in

$$\widetilde{\mu_2}^{-1}(0) \subset \widetilde{\mu_G}^{-1}(\mathcal{O}^{\mathbb{C}}_{\lambda_0}) \times \mathcal{O}^{\mathbb{C}}_{\lambda_0} \tag{1.18}$$

satisfy the condition

$$\widetilde{\mu_G}(\lambda_p) = -\lambda_p \tag{1.19}$$

As before, we can conclude, that $\widetilde{\mu_2}^{-1}(0)/G^{\mathbb{C}} \cong \mathcal{P}\mathcal{O}^{\mathbb{C}}_{\lambda_0}$.

Let us now define the map

$$F : T^*P \times M_{\lambda_0} \longrightarrow T^*P \times \mathcal{O}^{\mathbb{C}}_{\lambda_0}$$

by the formula

$$F(\lambda_p, x) = (\lambda_p, \mu_{G^{\mathbb{C}}}(x)) .$$

Since $\mu_{G^{\mathbb{C}}}$ is a symplectic diffeomorphism, so is F. In addition F is equivariant with respect to the $G^{\mathbb{C}}$-action. From 1.16, 1.17, 1.18, 1.19 we see, that the restriction

$$F : \widetilde{\mu_1}^{-1}(0) \longrightarrow \widetilde{\mu_2}^{-1}(0)$$

is still a diffeomorphism, and therefore the induced map

$$\widetilde{F} : (\mathcal{P}M_{\lambda_0} \ , \ \omega_{\lambda_0}) \longrightarrow (\mathcal{P}\mathcal{O}^{\mathbb{C}}_{\lambda_0} \ , \ \omega_{\mathcal{P}KK}) \tag{1.20}$$

is a symplectomorphism.

The second part of the proof is comparison between the symplectic quotients $\mathcal{P}M_{\lambda_0}$ and T^*P/B_{λ_0}. These spaces are both symplectic quotients of T^*P with respect to the action of the group B_{λ_0}. Choose a connection A on the principal bundle

$$\begin{array}{ccc} B_{\lambda_0} & \longrightarrow & P \\ & & \downarrow \\ & & M \end{array}$$

For each point $p \in P$ this gives a map $A_p : T_pP \to \mathfrak{b}_{\lambda_0}$, and dually an injection $A_p^* : \mathfrak{b}^*_{\lambda_0} \to T_p^*P$. We can then compose A_p^* with the $\widetilde{\mu_B}$ to get the mapping

$$A^* \circ \widetilde{\mu_B} : T^*P \longrightarrow T^*P .$$

For every $\lambda_p \in T^*P$ we have

$$(A^*\widetilde{\mu_B}(\lambda_p)) \in \widetilde{\mu_B}^{-1}(\widetilde{\mu_B}(\lambda_p))\,. \tag{1.21}$$

Indeed, let $\xi \in \mathfrak{b}_{\lambda_0}$ be an arbitrary element and ξ_B the vector field on P generated by the infinitesimal action of ξ. Then we get from the expression for the moment map on the cotangent bundle:

$$\langle \widetilde{\mu_B}(A_p^*\widetilde{\mu_B})(\lambda_p)\ ,\ \xi\rangle = \langle \widetilde{\mu_B}(\lambda_p)\ ,\ A_p(\xi_B)\rangle\,,$$

and since the connection is equal to the canonical Maurer-Cartan form when restricted to a fibre of the bundle, we finally get

$$\langle \widetilde{\mu_B}(A_p^*\widetilde{\mu_B})(\lambda_p)\ ,\ \xi\rangle = \langle \widetilde{\mu_B}(\lambda_p)\ ,\ \xi\rangle\,,$$

which proves 1.21.

The moment map μ of the natural action of a group G on a cotangent bundle T^*M is linear on each fibre T_xM. This is immediately seen from the expression

$$\langle \mu(\lambda_x), \xi\rangle = \langle \lambda_x, (\xi_G)_x\rangle$$

which holds for every $\xi \in \mathfrak{g}$. Therefore the space $\widetilde{\mu_B}^{-1}(\lambda_0) \cap T_p^*P$ is an affine space modelled on the vector subspace $\widetilde{\mu_B}^{-1}(0) \cap T_p^*P$, and 1.21 allows us to define the map

$$R : T^*P \longrightarrow \widetilde{\mu_B}^{-1}(0)$$

by the formula

$$R(\lambda_p) = \lambda_p - A_p^*(\widetilde{\mu_B}(\lambda_p)).$$

This map is B_{λ_0}-equivariant since the connection A and the moment map $\widetilde{\mu_B}$ are. So, after restricting to

$$R : \widetilde{\mu_B}^{-1}(\lambda_0) \longrightarrow \widetilde{\mu_B}^{-1}(0)$$

we get the induced map

$$\widetilde{R} : \mathcal{P}M_{\lambda_0} \longrightarrow T^*(P/B_{\lambda_0})\,, \tag{1.22}$$

which is a diffeomorphism.

To complete the proof of the proposition, we have to construct the 2-form β_{λ_0} on T^*P, such that $R^*(\omega_{can} + \beta_{\lambda_0}) = \omega_{can}$. Denote the composition $\lambda_0 \circ A_p$ by A_{λ_0}. This is a 1-form on P. Then for an arbitrary pair X, Y of vector fields on T^*P and after restricting on $\widetilde{\mu_B}^{-1}(\lambda_0)$, we get

$$R^*(\omega_{can})_{\lambda_p}(X,Y) = (\omega_{can})_{\lambda_p - A^*(\lambda_0)}(R_*X, R_*Y)\,.$$

Denoting the tautological 1-form on T^*P by α and again using the fact $\omega_{can} = d\alpha$, we get

$$\begin{aligned}(R^*\omega_{can})(X,Y) &= (R_*X)\langle\lambda_p\ ,\ \pi(R_*Y)\rangle - (R_*Y)\langle\lambda_p\ ,\ \pi(R_*Y)\rangle \\ &+ \langle\lambda_p\ ,\ \pi[R_*X, R_*Y]\rangle \\ &- (R_*X)\langle\lambda_0\ ,\ A_p(\pi(R_*Y))\rangle + \langle\lambda_0\ ,\ A_p(\pi(R_*Y))\rangle \\ &- \langle\lambda_0\ ,\ A_p(\pi[R_*X, R_*Y])\rangle\ .\end{aligned} \tag{1.23}$$

By π we denoted the derivative of the natural projection $\pi : T^*P \to P$.

Using similar computations as in the proof of proposition 2, it is not difficult to see, that the first two lines in the right-hand side of 1.23 are equal to $\omega_{can}(X,Y)$. The last two lines are clearly equal to $-R^*(dA_{\lambda_0})(x,Y)$.

The composition map

$$\widetilde{S} = \widetilde{F}^{-1} \circ \widetilde{R} : (\mathcal{PO}^{\mathbb{C}}_{\lambda_0}\ ,\ \omega_{\mathcal{P}KK}) \longrightarrow (T^*(P/B_{\lambda_0})\ ,\ \omega_{can} + \beta_{\lambda_0})$$

is then the searched-for symplectic diffeomorphism and the proposition is proved.

□

Chapter 2

Integrable systems on moduli spaces of parabolic bundles

In his paper [Hi 1] Hitchin constructs Liouville integrable systems on the cotangent bundle $T^*\mathcal{M}$ where $\mathcal{M}$ is the moduli space of holomorphic principal $G^{\mathbb{C}}$-bundles over a Riemann surface C. In this chapter our aim is to describe the analogous construction on the cotangent bundle $T^*\mathcal{M}_{par}$ of the moduli space of parabolic bundles over C. The group $G^{\mathbb{C}}$ will be classical semi-simple Lie group.

Our approach to this task is the following. First we describe the cotangent bundle $T^*\mathcal{M}_D$ of the moduli spaces of bundles over C with framings over the points p_i of a divisor D. There is a natural action of the group $G_D^{\mathbb{C}} \cong \prod_{i=1}^{degD} G_i^{\mathbb{C}}$ on $\mathcal{M}_D$ which we can lift to a symplectic action on $T^*\mathcal{M}_D$. There are many possible ways of taking the symplectic quotient with respect to this action.

Here we concentrate on the following cases. Let first B_D denote the subgroup of $G_D^{\mathbb{C}}$ such that for each $i = 1, \dots, deg(D)$ the group B_D is a parabolic subgroup of $G_i^{\mathbb{C}}$. Then B_D acts symplecticaly on $T^*\mathcal{M}_D$. We denote the corresponding moment map by $\mu_B : T^*\mathcal{M}_D \to (\mathfrak{b}_D)^*$. Then, as we shall see, the symplectic quotient $(\mu_B^{-1}(0)/B_D, \omega)$ is the space $(T^*\mathcal{M}_{par}, \omega_{can})$, where $\mathcal{M}_{par}$ denotes a large open cell in the moduli space of parabolic bundles over C having the elements $p_i \in D$ as marked points. Moduli spaces of parabolic bundles were constructed and studied by many authors, the first being Mehta and Seshadri with their paper [Me-Se]. Cotangent bundles $T^*\mathcal{M}_{par}$ are dense open sets in compact spaces called moduli spaces of parabolic Higgs bundles. These are a generalisation of Hitchin's moduli spaces of Higgs bundles in the same way as moduli spaces of paraboilic bundles are a generalisation of moduli spaces of holomorphic bundles. Spaces of parabolic Higgs bundles were studied by Boden in [Bo], Konno in [Ko] and others.

Consider now the action of the whole group $G_D^{\mathbb{C}}$ on $T^*\mathcal{M}_D$ giving the moment map $\mu_D : T^*\mathcal{M}_D \to (\mathfrak{g}_D^{\mathbb{C}})^*$. Choose a Cartan subalgebra $\mathfrak{h}_i$ in each of ${\mathfrak{g}^{\mathbb{C}}}_i$ and let

$\lambda_D = (\lambda_1, \ldots, \lambda_{deg(D)}) \in \bigoplus_{i=1}^{deg(D)} (\mathfrak{h}_i)^*$ be such that each $\lambda_i \in \mathfrak{h}_i$ is a regular element. Then we can form symplectic quotient $(\mu_D^{-1}(0)/H_D, \omega_{\mathcal{M}KK})$ where $Lie(H_D) = \bigoplus_{i=1}^{deg(D)} \mathfrak{h}_i$ and $\omega_{\mathcal{M}KK}$ denotes the induced symplectic form. It turns out that the space $\mu_D^{-1}(0)/H_D$ is diffeomorphic to $T^*\mathcal{M}_{par}$, but the symplectic form $\omega_{\mathcal{M}KK}$ is different from the canonical one.

Let $\mathcal{S}'$ denote the set of all $\lambda_D \in \bigoplus_{i=1}^{deg(D)} (\mathfrak{h}_i)^*$ having regular components λ_i and let $\mathcal{S} = \mathcal{S}' \cup \{0\} \subset \bigoplus_{i=1}^{deg(D)} (\mathfrak{h}_i)^*$. The main result of this chapter is the following theorem

Theorem 3 *Let $T^*\mathcal{M}_{par}$ be the cotangent bundle over the moduli space of parabolic bundles. There is a familly of complex structures I_{λ_D} on $T^*\mathcal{M}_{par}$ parametrised by $\mathcal{S}$ and a holomorphic symplectic structure ω_{λ_D} for each of these complex structures. Every space $(T^*\mathcal{M}_{par}, \omega_{\lambda_D})$ is equipped with an integrable system, i.e. there exist systems of holomorphic functions*

$$f^i_{\lambda_D} : (T^*\mathcal{M}_{par}, I_{\lambda_D}) \longrightarrow \mathbb{C} \quad i = 1, \ldots, dim(\mathcal{M}_{par})$$

which Poisson commute with respect to ω_{λ_D} and are functionally independent.

We note that the space $(T^*\mathcal{M}_{par}, I_{\lambda_D})$ where $\lambda_D = 0 \in \bigoplus_{i=1}^{deg(D)} (\mathfrak{h}_i)^*$ has the canonical form ω_{can} as the corresponding holomorphic symplectic structure. It is an exceptional fibre in our family in the sense that will be explained later.

Symplectic spaces $(T^*\mathcal{M}_{par}, \omega_{can})$ and $(T^*\mathcal{M}_{par}, \omega_{\lambda_D})$ are described in section 2.2. The systems of Poissson commuting functions $f^i_{\lambda_D}$ on these spaces are constructed in section 2.3 and finally their functional indepencdence is established in section 2.4. The functional independence will actually be proved only for the case $G^{\mathbb{C}} = SL(n;\mathbb{C})$. The cases of other classical groups could be covered using the approach of Hitchin in [Hi 1]. In section 2.4 spectral curves for our systems are discussed. Recall that spectral curves lie in the canonical bundle $T^*C = K$. It is shown that the spectral curve S_{λ_D} for the integrable system on $(T^*\mathcal{M}_{par}, \omega_{\lambda_D})$ has fixed intersection with the fibres of K over the points $p_i \in D$. In the case of $(T^*\mathcal{M}_{par}, \omega_{can})$ the curve S_0 intersects K_{p_i} at the point $0 \in K_{p_i}$, thus having ramification points of the maximum degree at the elements of D. At the end of this section the appropriate Jacobian tori are described as well as the way in which we recover the parabolic structure at $p_i \in D$ from the curve S and its Jacobian.

2.1 Moduli spaces of framed bundles

In this section we will describe the space $\mathcal{M}_D$ whose elements are isomorphism classes of complex structures on a principal bundle $P \to C$ together with framings over the points of a divisor D in C. The cotangent bundle $T^*\mathcal{M}_D$ with its canonical symplectic structure ω_{can} is going to be the starting point of our further constructions in this chapter. Later in the text we shall see that symplectic manifolds $(T^*\mathcal{M}_D\ ,\ \omega_{can})$ possess Hamiltonian actions of certain groups, and the symplectic quotients with respect to these actions will lead to symplectic spaces equipped with integrable systems.

2.1.1

Let C be a Riemann surface and let the divisor D on C consist of points $p_1, \ldots, p_m$ all with multiplicity 1. Let $P \longrightarrow C$ be a trivial principal bundle, whose fibre is a semi-simple complex Lie group $G^{\mathbb{C}}$. Denote by $\mathcal{G}^{\mathbb{C}}$ the group of gauge transformations of P and by A a holomorphic structure on this bundle .

Definition 4 *A holomorphic structure on P framed over D is a holomorphic structure A on P together with an isomorphism of $G^{\mathbb{C}}$-spaces*

$$\phi : P_{/D} \longrightarrow \prod_{i=1}^{deg(D)} G_i^{\mathbb{C}}\ .$$

Two framed holomorphic structures (A_1, ϕ_1) and (A_2, ϕ_2) are equivalent if there exists a gauge transformation $g \in \mathcal{G}^{\mathbb{C}}$ such that the conditions

$$g(A_1) = A_2$$

and

$$g \cdot \phi_1 = \phi_2$$

are satisfied.

Our starting object will be the space of orbits with respect to the action of $\mathcal{G}^{\mathbb{C}}$ on the space of framed holomorphic structures on P. In order to insure a good quotient space we have to introduce a suitable notion of stability.

Definition 5 *Let A be a holomorphic structure on $P \to C$ and let $\delta = deg(D)$. Then A is δ-stable if for every proper holomorphic sub-bundle F of adP we have*

$$\frac{\mathrm{deg}F}{\mathrm{rk}F} < \frac{\mathrm{deg}(adP)}{\mathrm{rk}(adP)} + \delta(\frac{1}{\mathrm{rk}F} - \frac{1}{\mathrm{rk}(adP)})\ .$$

Denote the space of δ-stable holomorphic structures on P with framings over the points of D by $\mathcal{B}^{\delta s}$. The group $\mathcal{G}^{\mathbb{C}}$ preserves $\mathcal{B}^{\delta s}$ and the quotient space $\mathcal{B}^{\delta s}/\mathcal{G}^{\mathbb{C}} = \mathcal{M}_D$ is a smooth finite dimensional manifold. This space was constructed and studied by Seshadri in [Se 1].

Let $p_i \in D$ be a marked point. The choice of a framing ϕ at p_i is equivalent to the choice of an element $g \in G^{\mathbb{C}}$. Choose an arbitrary point $pt \in P_{p_i}$ and let $\phi(pt) = g$ Then because of the $G^{\mathbb{C}}$-equivariance of ϕ, we have $\phi(h \cdot pt) = hg$ for every $h \in G^{\mathbb{C}}$, which completely determines the map ϕ. So the choice of a point pt in P_{p_i} yields a correspondence Ψ between the framings and the group $G^{\mathbb{C}}$ given by $\Psi(\phi) = \phi(pt)$.

Example 1 *Let $E \to C$ be a vector bundle of rank n over the curve C, and let $P \to C$ be the frame bundle of E having $Gl(n;\mathbb{C})$ as the fibre. Then the choice of $\phi : P_{p_i} \to Gl(n;\mathbb{C})$ obviously corresponds to the choice of framing $\varphi : E_{p_i} \to \mathbb{C}^n$. This choice of basis gives a concrete matrix representation, i.e. an element of $Gl(n;\mathbb{C})$ to the frame pt of E_{p_i}.*

Let $adP \to C$ be the vector bundle associated to the principal bundle P via the coadjoint representation of the group $G^{\mathbb{C}}$ in its Lie algebra $(\mathfrak{g}^{\mathbb{C}})^*$, and let

$$\overline{\partial}_A : \Omega^0(C; adP) \longrightarrow \Omega^{0,1}(C : adP)$$

be the partial covariant derivative on the bundle adP corresponding to the holomorphic structure A on P. Denote by $\mathcal{A}$ the space of all such $\overline{\partial}_A$. The space $\mathcal{A}$ is an affine space over the infinite-dimensional vector space $\Omega^{0,1}(C\,; ad\, P)$. The group of gauge transformations $\mathcal{G}^{\mathbb{C}}$ acts on $\mathcal{A}$ by conjugations:

$$g \cdot \overline{\partial}_A = g \circ \overline{\partial}_A \circ g^{-1},$$

where $\circ$ denotes the composition of operators.

Let now $\mathcal{G}_D^{\mathbb{C}}$ denote the subgroup of $\mathcal{G}^{\mathbb{C}}$ given by

$$\mathcal{G}_D^{\mathbb{C}} = \{g \in Aut(P);\ g(p_i) = id \quad \text{for every}\, p_i \in D\}\ ,$$

and let $\mathcal{A}^{\delta s}$ denote the subspace of $\mathcal{A}$ consisting of those operators $\overline{\partial}_A$ for which the corresponding holomorphic structure is δ-stable. Group $\mathcal{G}_D^{\mathbb{C}}$ preserves $\mathcal{A}^{\delta s}$ and we clearly have

$$\mathcal{M}_D = \mathcal{A}^{\delta s}/\mathcal{G}_D^{\mathbb{C}}\ .$$

Here we view the framings as fixed, the group $\mathcal{G}_D^{\mathbb{C}}$ being their stabiliser within the group of gauge transformations.

Next we are going to describe the cotangent bundle $T^*\mathcal{M}_D$. Every cotangent bundle carries the canonical symplectic structure, and we are going to make use of this fact in our description of $T^*\mathcal{M}_D$. We have seen in section 1.2.1 that a certain symplectic quotient of an arbitrary cotangent bundle T^*N is very easily describable. Namely, if the symplectic action of a group G on T^*N is the cotangent lifting of an action of G

on the base space N, then $\lambda^{-1}(0)/G = T^*(N/G)$, where λ denotes the moment map of the lifted action.

In our case the starting point will be the action of $\mathcal{G}_D^{\mathbb{C}}$ on $\mathcal{A}^{\delta s}$. Lift this action on the cotangent bundle $T^*\mathcal{A}^{\delta s}$ and let

$$\mu : T^*\mathcal{A}^{\delta s} \longrightarrow Lie(\mathcal{G}_D^{\mathbb{C}}) \tag{2.1}$$

be the moment map of this action. Then we can form the symplectic quotient

$$T^*\mathcal{M}_D = \mu^{-1}(0)/\mathcal{G}_D^{\mathbb{C}}, \tag{2.2}$$

where $T^*\mathcal{M}_D$ denotes the cotangent bundle of the moduli space of stable framed holomorphic structures on the bundle $P \to C$. This description of $T^*\mathcal{M}_D$ turns out to be very convenient because of the relative simplicity of the cotangent bundle $T^*\mathcal{A}^{\delta s}$.

We have already mentioned that $\mathcal{A}$ is an affine space with the underlying vector space $\Omega^{0,1}(C;\ adP)$. Let $(adP)^*$ denote the bundle dual to adP, and let $\langle\ ,\ \rangle_x$ be the natural fibrewise pairing. Then we can define the pairing of spaces $\Omega^{0,1}(C;\ adP)$ and $\Omega^{1,0}(C;\ (adP)^*)$ by

$$\langle \Phi, \Psi \rangle = \int_C \langle \Phi, \Psi \rangle_x\ . \tag{2.3}$$

Denote by $\mathcal{K}$ the Killing form on $\mathfrak{g}^{\mathbb{C}}$. Since $\mathfrak{g}^{\mathbb{C}}$ is a semi-simple Lie algebra, the Killing form is non-degenerate and can be used to identify the bundles adP and $(adP)^*$. So the pairing 2.3 will be replaced by the pairing of the spaces $\Omega^{0,1}(C;\ adP)$ and $\Omega^{1,0}(C;\ adP) \cong \Omega^0(C;\ adP \otimes K)$ given by

$$\langle \Phi, \Psi \rangle = \int_C \mathcal{K}(\Phi \wedge \Psi)\ ,$$

where Φ is an element of the first and Ψ of the second of the above spaces. Here $K \to C$ denotes the canonical bundle. The identification of the bundle $(adP)^*$ with the bundle adP will often be assumed tacitly from now on.

The function space $\Omega^0(C;\ ad\,P \otimes K)$ is of course not complete, and it will indeed turn out to be too small for our purposes. Since the Riemannian surface C is compact, we can view the space $\Omega^{0,1}(C;\ ad\,P)$ as the space of test functions. Its natural dual space will then be the space $\mathcal{D}(C;\ ad\,P \otimes K)$ of distributions on C with values in the bundle $ad\,P \otimes K$. This space contains the space of sections $\Omega^0(C;\ ad\,P \otimes K)$ in the natural way. From this we finally get

$$T^*\mathcal{A} \cong \mathcal{A} \times \mathcal{D}(C;\ ad\,P \otimes K).$$

Let $K(D)$ be the line bundle $K \otimes [D]$.

Proposition 5 *Let $[A] \in \mathcal{M}_D$ be a gauge-equivalence class of framed holomorphic structures on P containing A. Then the cotangent space $T^*_{[A]}\mathcal{M}_D$ is isomorphic to the space $H^0_A(C : adP \otimes K(D))$ of global A-holomorphic sections of the vector bundle adP twisted by the line bundle $K(D)$.*

Proof: In order to describe the cotangent space $T^*_{[A]}\mathcal{M}_D$ it is enough to know the form of the subspace $\mu^{-1}(0) \subset T^*\mathcal{A} = \mathcal{A} \times \mathcal{D}(C;\, ad\,P \otimes K)$ the action of $\mathcal{G}^{\mathbb{C}}_D$ on $T^*\mathcal{A}$ being the natural lifting of the action of $\mathcal{G}^{\mathbb{C}}_D$ on the base space $\mathcal{A}$.

After giving a convenient expression of the moment map μ as in [Hi 1], we are going to describe the behaviour of an element $(A, \Phi) \in \mathcal{A} \times \mathcal{D}(C; adP \otimes K)$ first in the neighbourhood of an ordinary point $p \in C \backslash supp(D)$ and then in the neighbourhood of a marked point $p \in D$.

Let $(A, \Phi) \in T^*\mathcal{A} = \mathcal{A} \times \mathcal{D}(C;\, ad\,P \otimes K)$ and $\widetilde{\psi} \in (Lie(\mathcal{G}^{\mathbb{C}}_D))^*$ be arbitrary elements. Then, as we have seen in section 1.2

$$\langle \mu((A, \Phi))\ ,\ \psi \rangle = f_\psi((A, \Phi)).$$

Here ψ denotes the element in $Lie(\mathcal{G}^{\mathbb{C}}_D)$ dual to $\widetilde{\psi}$ with respect to the pairing

$$\langle \widetilde{\psi}\ ,\ \psi \rangle = \int_C \mathcal{K}(\widetilde{\psi}\ ,\ \psi) \quad ,$$

and $f_\psi\ : T^*\mathcal{A} \to \mathbb{C}$ is the Hamiltonian function corresponding to the vector field $\widetilde{X_\psi}$ on $T^*\mathcal{A}$ generated by the infinitesimal action of ψ. From this we see that (A, Φ) lies in the subspace $\mu^{-1}(0)$ if and only if the Hamiltonian functions f_ψ corresponding to all $\psi \in Lie(\mathcal{G}^{\mathbb{C}}_D)$ take value zero at the point (A, Φ).

Since $\mathcal{G}^{\mathbb{C}}_D$ acts on $T^*\mathcal{A}$ by pull-backs, the following holds for the Hamiltonian functions.

$$f_\psi(A, \Phi) = (\imath(\widetilde{X_\psi})\alpha)(A, \Phi),$$

as was shown in section 1.2. Here α stands for the tautological form on $T^*\mathcal{A}$. By the definition of α we then have

$$f_\psi(A, \Phi) = \langle \widetilde{X_\psi}\ ,\ \Phi \rangle = \int_C \mathcal{K}(X_\psi\ ,\ \Phi),$$

where X_ψ is the vector field on $\mathcal{A}$ whose lifting on $T^*\mathcal{A}$ is $\widetilde{X_\psi}$. Let $g(t) : I \to \mathcal{G}^{\mathbb{C}}_D$ be a path such that $g(0) = id$ and $(\frac{d}{dt}g)(0) = \psi$. Than we have

$$X_\psi(A) = \frac{d}{dt}\,(g(t) \circ \overline{\partial}_A \circ g(t)^{-1})\ |_{t=0} = \overline{\partial}_A\ \psi. \tag{2.4}$$

This can easily be seen in a local trivialisation, where $\overline{\partial}_A$ has the expression $\overline{\partial}_A = \bar{\partial} + A$. Then

$$\frac{d}{dt}\left(g(t) \circ \overline{\partial}_A \circ g(t)^{-1}\right)\,|_{t=0} = [\bar{\partial} + A, \psi] = \bar{\partial}\psi + [A, \psi] = \overline{\partial}_A \psi$$

Formula 2.4 finally gives us the desired expression of the Hamiltonian functions

$$f_\psi(A, \Phi) = \int_C \mathcal{K}(\overline{\partial}_A\ \psi\ \wedge\ \Phi)\,.$$

So the point $(A, \Phi) \in T^*\mathcal{A} = \mathcal{A} \times \mathcal{D}(C; ad\, P \otimes K)$ lies in the preimage $\mu^{-1}(0)$ if and only if

$$\int_C \mathcal{K}(\overline{\partial}_A\ \psi\ \wedge\ \Phi) = 0$$

for every $\psi \in Lie(\mathcal{G}_D^{\mathbb{C}})$.

Let now $p \in C\backslash supp(D)$ be an arbitrary point, and let $U \subset C\backslash supp(D)$ be a neighbourhood of p. For every ψ such that $supp(\psi) \subset C\backslash supp(D)$ we get from Stokes' theorem

$$\int_C \mathcal{K}(\overline{\partial}_A\ \psi \wedge \Phi) = \int_C \mathcal{K}(\psi \cdot \overline{\partial}_A\ \Phi) = 0$$

Placing ourselves in a holomorphic local trivialisation of $P_{/U}$ this gives the condition

$$\bar{\partial}\Phi = 0$$

on U. The $\bar{\partial}$-regularity theorem ([G-H], page 380) then tells us that the distribution Φ is actually a function and hence a holomorphic function on U.

Let now U be a neighbourhood of a point p in the divisor D. In this case we get the condition

$$\int_C \mathcal{K}(\overline{\partial}_A\ \psi \wedge \Phi) = \psi(p),$$

since $\psi \in \mathcal{G}_D^{\mathbb{C}}$, and therefore $\psi(p) = 0$. This gives the equations of distributions

$$\bar{\partial}\ \Phi = a \cdot \delta(p) \quad ,$$

where $a \in \mathfrak{g}^{\mathbb{C}}$ is an arbitrary element and $\delta(p)$ denotes the Dirac function centred at p. From the Cauchy integral formula it is then readily seen that Φ is a meromorphic function on U with the first order pole in p having a as the residue at p.

We have proved that if (A, Φ) is an element of $\mu^{-1}(0)$ than it is an A-meromorphic section of the bundle $ad\,P \otimes K$ having simple poles at the points of D. The space of such meromorphic sections is naturally isomorphic to the space of holomorphic sections $H^0(C\ ;\ adP \otimes K(D))$ of the twisted bundle $adP \otimes K(D)$. The isomorphism is provided by tensoring the meromorphic sections by the section $\sigma \in H^0(C\ ;\ [D])$ having $[D]$ as the zero divisor. □

Remark 2 *Let the subspace $T^*_{sm}\mathcal{A} \subset T^*\mathcal{A}$ consist of the pairs (A, Φ), such that Φ is smooth almost everywhere. It follows immediately from the proof above that the moment map*

$$\mu : T^*_{sm}\mathcal{A} \longrightarrow Lie(\mathcal{G}^{\mathbb{C}}_D)^*$$

is given by the formula

$$\mu(A, \Phi) = \overline{\partial}_A \Phi - \sum_{i=1}^{deg(D)} Res(\Phi)_{p_i} \cdot \delta(p_i)\ ,$$

where $Res(\Phi)_{p_i}$ is understood to be the residue in the sense of distributions as described e.g. in chapter 3 of [G-H].

It should not be forgotten that in the above formula the identification of $Lie(\mathcal{G}^{\mathbb{C}}_D)^*$ with $Lie(\mathcal{G}^{\mathbb{C}}_D)$ via the Killing form $\mathcal{K}$ is implied. More correctly we should write

$$\mu(A, \Phi) = \mathcal{K}(\quad \cdot \quad ,\quad \overline{\partial}_A \Phi - \sum_{i=1}^{deg(D)} Res(\Phi)_{p_i} \cdot \delta(p_i))\ .$$

2.1.2

Following Hitchin's recipe, we are going now to produce a set of functions on the cotangent bundle $T^*\mathcal{M}_D$ which will Poisson commute with respect to the canonical symplectic structure on $T^*\mathcal{M}_D$. The representation of $T^*\mathcal{M}_D$ as a symplectic quotient (2.2) is the key ingredient.

Let $(q_1, \ldots, q_r)$ be a basis of invariant polynomials of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$. Here r is the rank of $\mathfrak{g}^{\mathbb{C}}$. Denote by d_i the degrees of q_i and define the map

$$\widetilde{\mathbf{H}} : \mathcal{A} \times \Omega^0(C\ ;\ adP \otimes K(D)) \longrightarrow \bigoplus_{i=1}^{r} \Omega(C\ ;\ K(D)^{d_i})$$

by the formula

$$\widetilde{\mathbf{H}}(A, \Phi) = (q_1(\Phi), \ldots, q_r(\Phi))$$

On the previous page we have seen that the subspace $\mu^{-1}(0) \subset T^*\mathcal{A}$ is the space of pairs $\{(A, \Phi)\ ; A \in \mathcal{A}, , \Phi \in H^0_A(c; adP \otimes K(D))\}$

Restrict now $\widetilde{\mathbf{H}}$ on the subspace $\mu^{-1}(0) \cap T^*\mathcal{A}^{\delta s}$. The map $\widetilde{\mathbf{H}}$ is invariant under the action of the gauge group $\mathcal{G}^{\mathbb{C}}_D$, so it induces the map

$$\mathbf{H} : T^*\mathcal{M}_D \longrightarrow \bigoplus_{i=1}^{r} H^0(C\ ;\ K(D)^{d_i}) \quad . \tag{2.5}$$

The components of the map $\mathbf{H}$ will be the functions that we are looking for.

Choose a basis $\alpha_i^1, \ldots, \alpha_i^{k_i}$ of the dual space $H^0(C\ ;\ K(D)^{d_i})^*$ and define the functions

$$f_{i,j} : T^*\mathcal{M}_D \longrightarrow \mathbb{C}$$

by

$$f_{i,j}(\Phi) = \langle \mathbf{H}(\Phi), \alpha_i^j \rangle \tag{2.6}$$

Proposition 6 *The functions $f_{i,j}$ Poisson-commute with respect to the canonical symplectic structure on $T^*\mathcal{M}_D$.*

Proof: The spaces $H^0(C\ ;\ K(D)^{d_i})^*$ are isomorphic to $H^1(C\ ;\ K^{1-d_i} \otimes [D]^{-d_i})$ by Serre duality. For each α_i^j choose a representative $(0,1)$-form $\beta_i^j \in \Omega^{0,1}(C\ ;\ K^{1-d_i} \otimes [D]^{-d_i})$. The definition of the functions $f_{i,j,}$ then has the form

$$f_{i,j}(\Phi) = \int_C q_i(\Phi) \cdot \beta_i^j \quad .$$

Using this formula, we can define the functions $\tilde{f}_{i,j}$ on the space $T^*\mathcal{A}$. Obviously, the functions $f_{i,j}$ are induced from the functions $\tilde{f}_{i,j}$ in the sense of proposition 1 in section 1.2.1. The functions $\tilde{f}_{i,j}$ depend only on the cotangent directions and are independent of the directions on the base space $\mathcal{A}$. Therefore they Poisson commute with respect to the canonical symplectic form on $T^*\mathcal{A}$. Since $f_{i,j}$ are induced by $\tilde{f}_{i,j}$ they Poisson commute on $T^*\mathcal{M}_D$ as was shown in proposition 1.

2.1.3

In this subsection we are going to compute the dimensions of the spaces $\mathcal{M}_D$ and $\bigoplus_{i=1}^r H^0(C; K(D)^{d_i})$. The calculations are going to be straightforward applications of Riemann-Roch theorem.

Choose a holomorphic structure A on $P \to C$. Riemann-Roch theorem for the holomorphic vector bundles of rank n over the Riemannian surface C of genus g combined with the Serre duality gives

$$h^0(C;\ adP \otimes K(D)) - h^0(C;\ adP \otimes [D]^*) = c_1(K(D)^n) + n\ (1-g)\ .$$

A straightforward calculation shows that δ-stability of a holomorphic bundle $P_A \to C$ implies that $H^0(C; adP \otimes [D]^*) = 0$ for every divisor D with $\deg(D) = \delta$. (For the proof see Lemma 4.6 in [Ma].) On the right-hand side we then have

$$c_1(K(D)^n) + n(1-g) = n(g-1) + n \cdot deg(D)\ .$$

From proposition 5 we then finally get

$$dim\ \mathcal{M}_D = n(g-1) + n \cdot deg(D). \tag{2.7}$$

Riemann-Roch theorem for line bundle $K(D)^{d_i}$ on the curve C of genus g has the form

$$h^0(C;\ K(D)^{d_i}) = (2d_i - 1)\ (g-1) + d_i \cdot deg(D) \quad .$$

The degrees d_i of the invariant functions on $\mathfrak{g}^{\mathbb{C}}$ and the dimension n of $\mathfrak{g}^{\mathbb{C}}$ are related by the equation

$$\sum_{i=1}^{r}(2d_i - 1) = n$$

proved e.g. in [Hu 1]. The summation then gives us

$$\sum_{i=1}^{r} h^0(C;\ K(D)^{d_i}) = n\ (g-1) + b \cdot deg(D)\ . \tag{2.8}$$

Here $b = 1/2(n+r)$ denotes the dimension of a Borel sub-algebra $\mathfrak{b} \subset \mathfrak{g}^{\mathbb{C}}$.

Comparing numbers 2.7 and 2.8 we see, that the functions $f_{i,j}$ even though Poisson commuting do not constitute an integrable system on the space $T^*\mathcal{M}_D$, since their number is less than the dimension of $\mathcal{M}_D$.

2.2 Symplectic quotients of $T^*\mathcal{M}_D$

In the situation, when one has a symplectic space and a "large" number of Poisson-commuting functions which is still too small to form an integrable system, it is natural to try to find a symplectic sub-manifold or some kind of "symplectic slice" in the original space on which the reduced functions might form a completely integrable system. One of the appropriate tools to use in such situations is the symplectic quotient. Of course the action of the group yielding the symplectic quotient must keep the integrals invariant. In our case the integrals $f_{i,j}$ on $T^*\mathcal{M}_D$ come from the functions $\tilde{f}_{i,j}$ on $T^*\mathcal{A}$ which are invariant under the action of the full gauge group $\mathcal{G}^{\mathbb{C}}$. Since we used a subgroup $\mathcal{G}_D^{\mathbb{C}} \subset \mathcal{G}^{\mathbb{C}}$ to form the symplectic quotient $T^*\mathcal{M}_D$, we can still use the residual group $\mathcal{G}^{\mathbb{C}}/\mathcal{G}_D^{\mathbb{C}} \cong \prod_{i=1}^{deg(D)} G_i^{\mathbb{C}}$ and certain of its subgroups to form further symplectic quotients of $T^*\mathcal{M}_D$ on which the integrals $f_{i,j}$ will induce systems of Poisson-commuting functions. In the following subsection we are going to describe the action of the group $\mathcal{G}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}_D$ on $T^*\mathcal{M}_D$. Throughout the rest of this chapter (with the exception of proposition 10) the simbol $\mathcal{M}_D$ will denote the large open set in $\mathcal{M}_D$ on which the group $\mathcal{G}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}_D$ acts freely.

2.2.1

In terms of definition 4 the action of the group $\mathcal{G}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}_D$ on $\mathcal{M}_D$ is quite obvious. Let g be an element in $\prod_{i=1}^{deg(D)} G_i^{\mathbb{C}} \cong \mathcal{G}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}_D$ and $([A], \phi)$ an element in $\mathcal{M}_D$. Then the action of g on $([A], \phi)$ is defined by the formula

$$g \cdot ([A], \phi) = ([A], g \circ \phi)$$

Here $g = (g_1, \ldots, g_{deg(D)})$ on the right-hand side of the definition is understood as map from $\prod_{i=1}^{deg(D)} G_i^{\mathbb{C}}$ into itself given by

$$(g_1, \ldots, g_{deg(D)})((h_1, \ldots, h_{deg(D)})) = (g_1 h_1, \ldots, g_{deg(D)} h_{deg(D)}) ,$$

and $\circ$ denotes the composition of maps. The natural coadjoint lifting of this action gives the desired action on the space $T^*\mathcal{M}_D$.

In order to make use of the description of the space $T^*_{[A]}\mathcal{M}_D$ as $H^0(C;\ adP \otimes K(D))$, we need an expression of the cotangent action in terms of sections of bundles related to the associated bundle adP over the curve C. The associated bundle adP is the quotient $(P \times \mathfrak{g}^{\mathbb{C}})/G^{\mathbb{C}}$, where the action of the group $G^{\mathbb{C}}$ is given by the formula

$$g \cdot (p, \alpha) = (g \cdot p, Ad_g(\alpha)) .$$

From this we see that the framing $\phi : P_p \to G^{\mathbb{C}}$ of P over the point p will fix a particular Ad-action of $G^{\mathbb{C}}$ on the fibre adP_p of the associated bundle. This means, that if the framing ϕ equates an element a in the fibre adP_p with the element α in $\mathfrak{g}^{\mathbb{C}}$, then the framing $g \cdot \phi$ equates a with $Ad_g(\alpha)$. Therefore the elements $\psi \in \mathcal{G}^{\mathbb{C}}$ which are not equal to the identity at the point of D will not preserve the framings but act on them.

Let now $[A] \in \mathcal{M}_D$ and $\overline{\partial}_A$ be its representative in $\mathcal{A}$. Let $[g] \in \mathcal{G}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}_D$ with a representative $g \in \mathcal{G}^{\mathbb{C}}$. Then the action of $\mathcal{G}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}_D$ on $\mathcal{M}_D$ is defined by

$$[g] \cdot [A] = [g \circ \overline{\partial}_A \circ g^{-1}] \tag{2.9}$$

Obviously the definition is independent of the choice of the representatives. For any choice we are in the same $\mathcal{G}^{\mathbb{C}}$-orbit i.e. we have the same holomorphic structure. The class $[g]$ is defined by the values it assumes at the points of D, so we have indeed the action on the framings.

From now on we are going to denote the quotient group $\mathcal{G}^{\mathbb{C}}/\mathcal{G}_D^{\mathbb{C}} \cong \prod_{i=1}^{deg(D)} G_i^{\mathbb{C}}$ by $G_D^{\mathbb{C}}$. The symbol $[\psi]$ will stand for the element in the Lie algebra $Lie(G_D^{\mathbb{C}}) \cong Lie(\mathcal{G}^{\mathbb{C}}/\mathcal{G}_D^{\mathbb{C}})$ having $\psi \in Lie(\mathcal{G}^{\mathbb{C}})$ as a representative. Observe that the class $[\psi] \in Lie(\mathcal{G}^{\mathbb{C}}/\mathcal{G}_D^{\mathbb{C}})$ is uniquely defined by the values that any of its representatives $\psi \in \mathcal{G}^{\mathbb{C}}$ assumes at the points of D. Therefore we clearly have $Lie(G_D^{\mathbb{C}}) \cong \bigoplus_{i=1}^{deg(D)}(\mathfrak{g}_i^{\mathbb{C}})$. By $\Phi_{[A]}$ we are going to denote the covectors lying in $T^*_{[A]}\mathcal{M}_D \cong H^0(C;\ adP \otimes K(D))$. In the following proposition we are going to prove the formulas for the Hamiltonian functions and the moment map for the $G_D^{\mathbb{C}}$-action on $T^*\mathcal{M}_D$ which will be used in the constructions of symplectic quotients of $T^*\mathcal{M}_D$

Proposition 7 *Let $G_D^{\mathbb{C}}$ act on $T^*\mathcal{M}_D$ by the cotangent liftings of the action 2.9 defined above. Let $[\psi] \in Lie(G_D^{\mathbb{C}})$ be an arbitrary element, let*

$$f_{[\psi]} : T^*\mathcal{M}_D \longrightarrow \mathbb{C}$$

be the Hamiltonian function corresponding to the vector field $\widetilde{X}_{[\psi]}$ on $T^*\mathcal{M}_D$ generated by the infinitesimal action of ψ, and let

$$\mu_D : T^*\mathcal{M}_D \longrightarrow (Lie(G_D^{\mathbb{C}}))^* \cong \bigoplus_{i=1}^{deg(D)} (\mathfrak{g}_i^{\mathbb{C}})^*$$

be the moment map of $G_D^{\mathbb{C}}$-action. Then the Hamiltonian functions are given by the formula

$$f_{[\psi]}(\Phi_{[A]}) = \sum_{i=1}^{deg(D)} \mathcal{K}(\psi(p_i) \cdot Res_{p_i}\Phi_{[A]})\,, \tag{2.10}$$

and the moment map is given by

$$\mu_D(\Phi_{[A]}) = (Res_{p_1}\Phi_{[A]}, \ldots, Res_{p_{deg(D)}}\Phi_{[A]})\,. \tag{2.11}$$

Proof: Let $\overline{\partial}_A \in \mathcal{A}$ be a representative of the holomorphic structure $[A]$, and let $\psi \in [\psi]$ be a representative of $[\psi] \in G_D^{\mathbb{C}}$. Then it can be seen from the proof of proposition 1 that the Hamiltonian $f_{[\psi]}$ is given by the formula

$$f_{[\psi]}(\Phi_{[A]}) = \int_C \mathcal{K}(\overline{\partial}_A\psi \ \wedge \ \Phi_{[A]}) \tag{2.12}$$

We have to show that this definition is independent of the choice of representatives. Let $U_i^{\epsilon} \subset C$, $i = 1, \ldots, deg(D)$ be discs around the points $p_i \in D$ all of radius ϵ. Then by the Stokes' theorem we get

$$\int_C d\,\mathcal{K}(\psi \cdot \Phi_{[A]}) = \lim_{\epsilon \to 0} \int_{C \setminus \cup\, U_i^\epsilon} d\,\mathcal{K}(\psi \cdot \Phi_{[A]}) \tag{2.13}$$

$$= \sum_{i=1}^{deg(D)} \lim_{\epsilon \to 0} \int_{\partial U_i^\epsilon} \mathcal{K}(\psi \cdot \Phi_{[A]}) \tag{2.14}$$

On the other hand by Leibnitz rule and keeping the type decomposition in mind

$$\lim_{\epsilon \to 0} \int_{C \setminus \cup\, U_i^\epsilon} d\,\mathcal{K}(\psi \cdot \Phi_{[A]}) = \lim_{\epsilon \to 0} \int_{C \setminus \cup U_i^\epsilon} \mathcal{K}(\overline{\partial}_A \psi \cdot \Phi_{[A]}) + \lim_{\epsilon \to 0} \int_{C \setminus \cup\, U_i^\epsilon} \mathcal{K}(\psi \cdot \overline{\partial}_A \Phi_{[A]})$$

Since on $C \setminus \cup\, U_i^\epsilon$ we have $\overline{\partial}_A \Phi_{[A]} = 0$, the above two equations give us

$$\int_C \mathcal{K}(\overline{\partial}_A \psi \ \wedge\ \Phi_{[A]}) = \sum_{i=1}^{deg(D)} \lim_{\epsilon \to 0} \int_{\partial U_i^\epsilon} \mathcal{K}(\psi \cdot \Phi_{[A]}) \tag{2.15}$$

Let z be a local coordinate on a neighbourhood U_i^ϵ of the point $p_i \in D$ and choose a local trivialisation of adP over U_i. The section $\Phi_{[A]}$ has poles of degree one at the points of D, while ψ is smooth over U_i^ϵ. It can then be shown that $\bar{\partial}\mathcal{K}(\psi \cdot \Phi_{[A]})$ is an absolutely integrable function on U_i^ϵ. Therefore the Cauchy integral formula gives us

$$\lim_{\epsilon \to 0} \int_{\partial U_i^\epsilon} \mathcal{K}(\psi \cdot \Phi_{[A]}) = \mathcal{K}(\psi(p_i) \cdot Res_{p_i} \Phi_{[A]}) \tag{2.16}$$

Finally we can conclude from the equations 2.12, 2.15, and 2.16 that

$$f_{[\psi]}(\Phi_{[A]}) = \sum_{i=1}^{deg(D)} \mathcal{K}(\psi(p_i) \cdot Res_{p_i} \Phi_{[A]}) \tag{2.17}$$

From the expression for the moment map

$$\mu(x) = \sum_{i=1}^{n} f_{\xi_i}(x) \cdot \xi^i$$

explained in section 1.2, which gives us $\langle \mu_D(\Phi_{[A]}), [\psi] \rangle = f_{[\psi]}(\Phi_{[A]})$, and from the equation 2.17 , the formula

$$\mu_D(\Phi_{[A]}) = (Res_{p_1} \Phi_{[A]}, \ldots, Res_{p_{deg(D)}} \Phi_{[A]}) \tag{2.18}$$

follows immediately.

□

Remark 3 *As already mentioned in remark 2, we should actually write*

$$\mu_D(\Phi_{[A]}) = (\mathcal{K}(\,\cdot\,, Res_{p_1}\Phi_{[A]}), \ldots, \mathcal{K}(\,\cdot\,, Res_{p_{deg(D)}}\Phi_{[A]}))\ .$$

2.2.2

The following proposition describes the most obvious symplectic quotient of the space $T^*\mathcal{M}_D$.

Proposition 8 *Let $G_D^{\mathbb{C}}$ act on $T^*\mathcal{M}_D$ by the cotangent liftings of the action 2.9, and let*

$$\mu_D : T^*\mathcal{M}_D \longrightarrow \bigoplus_{i=1}^{deg(D)} (\mathfrak{g}_i^{\mathbb{C}})^*$$

be the moment map of this action. Then

$$\mu_D^{-1}(0)/{G^{\mathbb{C}}}_D = T^*\mathcal{M}\ .$$

Proof: Denote by $\tilde{\mu}$ the moment map of the action of the whole gauge group $\mathcal{G}^{\mathbb{C}}$ on $T^*\mathcal{A}$. Since $G_D^{\mathbb{C}} = \mathcal{G}^{\mathbb{C}}/\mathcal{G}_D^{\mathbb{C}}$, one way of obtaining the preimage $\mu_D^{-1}(0)$ is to restrict the preimage $\tilde{\mu}^{-1}(0)$ on the subspace $\mu^{-1}(0)$ where μ as before denotes the moment map of the $\mathcal{G}_D^{\mathbb{C}}$-action on $T^*\mathcal{A}$. Since the map $\tilde{\mu}$ is given by the formula $\tilde{\mu}(A,\Phi) = \overline{\partial}_A\Phi$, we see from remark 2 that the preimage $\mu_D^{-1}(0)$ consists of those elements $(A,\Phi) \in T^*\mathcal{A}$ which solve the pair of distribution equations

$$\overline{\partial}_A\Phi = 0 \tag{2.19}$$

$$\overline{\partial}_A\Phi - \sum_{i=1}^{deg(D)} Res(\Phi)_{p_i}\cdot\delta(p_i) = 0 \tag{2.20}$$

Since $\sum_{i=1}^{deg(D)} Res(\Phi)_{p_i}\cdot\delta(p_i)$ is equal to zero if and only if $Res(\Phi)_{p_i}$ is zero for every i, the solutions of the above system are pairs (A,Φ) such that $\overline{\partial}_A\Phi = 0$.

We can come to the same conclusion even more quickly using proposition 7 above. Since

$$\mu_D(\Phi_{[A]}) = (Res_{p_1}\Phi_{[A]}, \ldots, Res_{p_{deg(D)}}\Phi_{[A]})\ ,$$

the preimage $\mu_D^{-1}(0)$ will obviously consist of those meromorphic sections $\Phi_{[A]}$ of the bundle $adP \otimes K$ whose residues at the points $p_i \in D$ are all zero. These are precisely the holomorphic sections of $adP \otimes K$ with respect to the holomorphic structure $[A]$.

The space of holomorphic sections $H^0(C;\ adP_A \otimes K)$ is isomorphic to $T_A^*\mathcal{M}$. Since the G- action on the base space N of any cotangent bundle T^*N gives the symplectic quotient $\mu^{-1}(0)/G = T^*(N/G)$, and since $\mathcal{M}_D/G_D^{\mathbb{C}} = \mathcal{M}$ we get $\mu_D^{-1}(0)/G_D^{\mathbb{C}} = T^*\mathcal{M}$, which proves the proposition.

□

Let the complex semi-simple Lie group $G^{\mathbb{C}}$ act on itself by the left translations and let $\mu : T^*G^{\mathbb{C}} \to \mathfrak{g}^*$ be the moment map of the lifted $G^{\mathbb{C}}$-action on $T^*G^{\mathbb{C}}$. Then the symplectic quotient described in proposition 8 corresponds to the trivial symplectic quotient $\mu^{-1}(0)/G^{\mathbb{C}} = \{pt\}$. But, as we have seen in section 1.2, there exist more interesting quotients of $T^*G^{\mathbb{C}}$. One is obtained by taking in consideration the action of a Borel subgroup $B \in G^{\mathbb{C}}$ and the other by choosing some regular element $\lambda_0 \in (\mathfrak{g}^{\mathbb{C}})^*$ rather than zero for the value of the moment map μ. Below the symplectic quotients of $T^*\mathcal{M}_D$ analogous to these will be described.

Let as before $D \subset C$ be a reduced divisor consisting of points p_i, $i = 1, \ldots, deg(D)$. Choose for every i a Borel subgroup B_i in the copy $G_i^{\mathbb{C}}$ of the complex semi-simple group $G^{\mathbb{C}}$.

Definition 6 *A parabolic structure on P over the divisor D is a holomorphic structure A on P together with a $G^{\mathbb{C}}$ -equivariant map of $G^{\mathbb{C}}$-spaces*

$$\phi : P_{/D} \longrightarrow \prod_{i=1}^{deg(D)} G_i^{\mathbb{C}}/B_i \, .$$

Two parabolic structures (A_1, ϕ_1) and (A_2, ϕ_2) are equivalent if there exists a gauge transformation $g \in \mathcal{G}^{\mathbb{C}}$ such that the conditions

$$g(A_1) = A_2$$

and

$$g \cdot \phi_1 = \phi_2$$

are satisfied.

Denote the space of $\mathcal{G}^{\mathbb{C}}$-orbits of parabolic structures on the bundle P by $\mathcal{M}_{par}$. As in definition 4, it is clear that the parabolic structure ϕ is determined by the prescription $\phi(pt) = [g]$ of an element $[g] \in G_i^{\mathbb{C}}/B_i$ to some point $pt \in P_i$ for $i = 1, \ldots, deg(D)$.

Example 2 *Let again $E \to C$ be a holomorphic vector bundle of rank n over C, and let $P \to C$ be the associated $Gl(n;\mathbb{C})$-principal bundle of frames of E. We claim that the choice of ϕ and B_i corresponds to the choice of a flag $\mathcal{F}$:*

$$F_i^1 \subset F_i^2 \subset, \dots, \subset F_i^n = E_{p_i} .$$

Observe first that in presence of a hermitian metric on E the choice of a flag is equivalent to the choice of a sequence of one-dimensional linear subspaces $\{E_i^j\}_{j=1,\dots,n}$ given by $E_i^j = F_i^j/F_i^{j-1} \cong (F_i^{j-1})^\perp \subset F_i^j$, since then $F_i^j = span\{E_i^k\}_{k=1}^j$ for every j. Choose a frame $(e_i^1, \dots, e_i^n)$ of E_{p_i} such that $e_i^j \in E_i^j$ for all the indices i, j and denote this frame by $pt_i \in P_{p_i}$. Define

$$\phi(pt_i) = [id] \in G_i^{\mathbb{C}}/B_i \cong G/T$$

where $G \subset G^{\mathbb{C}}$ is the compact real form in $G^{\mathbb{C}}$ and T the maximal torus contained in B_i. Because of the $G^{\mathbb{C}}$-equivariance, this uniquely determines ϕ. From the equivariance it is also clear that $\phi^{-1}([id]) = \mathcal{F}$, so by the choice of ϕ the flag $\mathcal{F}$ becomes the B_i-orbit of $pt_i = (e_i^1, \dots, e_i^n)$.

Choose a Borel subgroup B_i in each of the complex groups $G_i^{\mathbb{C}}$, and denote the obtained subgroup of $G_D^{\mathbb{C}}$ by B_D. Then B_D is a subgroup of the quotient group $\mathcal{G}^{\mathbb{C}}/\mathcal{G}_D^{\mathbb{C}}$ consisting of classes $[\psi]$ of elements in $\mathcal{G}^{\mathbb{C}}$ which assume fixed values $\psi(p_i) = g_i$ at the points $p_i \in D$. In addition the values g_i lie in the subgroups B_i. Since the group $G_D^{\mathbb{C}}$ has a natural action on $T^*\mathcal{M}_D$, so has the subgroup B_D. Let $\mathfrak{b}_i$ be the Lie algebra of the group B_i, and let $\mathfrak{b}_i^*$ be its dual. Denote by $\mathfrak{n}_i$ the annihilator of $\mathfrak{b}_i^* \subset {\mathfrak{g}^{\mathbb{C}}}_i$ with respect to the dual pairing. In the identification of $(\mathfrak{b}_i)^*$ with $\mathfrak{b}_i$ using the Killing form, the subspace $\mathfrak{n}_i$ will become the $\mathcal{K}$-orthogonal complement of $\mathfrak{b}_i$. Let

$$\mu_B : T^*\mathcal{M}_D \longrightarrow \bigoplus_{i=1}^{deg(D)} (\mathfrak{b}_i)^*$$

be the moment map of the B_D-action.

Proposition 9 *Let B_D act in the natural way on the space $T^*\mathcal{M}_D$. Then the symplectic quotient $\mu_B^{-1}(0)/B_D$ is isomorphic to the space $T^*\mathcal{M}_{par}$,*

$$\mu_B^{-1}(0)/B_D \cong T^*\mathcal{M}_{par} .$$

The cotangent space $T^*_{[A]}\mathcal{M}_{par}$ can be identified with the space of meromorphic sections $\Phi_{[A]}$ of the bundle $adP \otimes K$ with respect to the holomorphic structure $[A]$ with first order poles at the points of D and with residues at the points p_i lying in the subspaces $\mathfrak{n}_i \subset \mathfrak{g}^{\mathbb{C}}$.

Proof: Again the symplectic quotient in question is a case of the cotangent reduction (the B_D-action on $T^*\mathcal{M}_D$ is the cotangent lifting of the action of B_D on the base space $\mathcal{M}_D$.) Therefore $\mu_B^{-1}(0)/B_D \cong T^*(\mathcal{M}_D/B_D)$.

Let $([A],\phi) \in \mathcal{M}_D$ be an arbitrary element. The B_D-action leaves $[A]$ invariant. Choose a point $pt \in P_{p_i}$ and suppose $\phi(pt) = g_i$. Then the B_i-orbit of ϕ are all the maps $\chi : P_{p_i} \to G_i^{\mathbb{C}}$ for which $\phi(pt)$ lies in the B_i orbit through g_i. So, the B_i-orbit of ϕ can be identified with the map $\tilde{\phi} : P_{p_i} \to G_i^{\mathbb{C}}/B_i$ such that $\tilde{\phi}(pt) = g_i \cdot B_i$. The situation is similar at each point $p_i \in D$, therefore we have $\mathcal{M}_D/B_D \cong \mathcal{M}_{par}$, and hence $\mu_B^{-1}(0)/B_D \cong T^*\mathcal{M}_{par}$ as claimed.

In order to obtain the description of $T^*_{[A]}\mathcal{M}_{par}$ in terms of the meromorphic sections, we just have to find the solutions of the equation

$$\mu_B(\Phi_{[A]}) = 0 \in \bigoplus_{i=1}^{deg(D)} (\mathfrak{b}_i)^* \, .$$

Clearly the Hamiltonian functions of the infinitesimal actions are given by the same formula as those in proposition 7, namely

$$f_{[\psi]}(\Phi_{[A]}) = \sum_{i=1}^{deg(D)} \mathcal{K}(\psi(p_i) \cdot Res_{p_i}\Phi_{[A]}) \tag{2.21}$$

Let $\{\xi_i^j\}$ be some bases of the Lie algebras $\mathfrak{b}_i$ and let $[\psi_i^j]$ be elements of $Lie(B_D)$ given by the representatives $\psi \in Lie(\mathcal{G}^{\mathbb{C}})$ satisfying the conditions

$$\psi_i^j(p_k) = \xi_i^j \, .$$

The elements $[\psi_i^j]$ constitute a basis for $Lie(B_D)$. Denote by $b = (1/2)(n+r)$ the dimension of a Borel sub-algebra in the n-dimensional Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of rank r. The map μ_B than has the expression

$$\mu_B(\Phi_{[A]}) = \sum_{i=1}^{deg(D)} \sum_{j=1}^{b} f_{[\psi_i^j]}(\Phi_{[A]}) \cdot [\psi_i^j]^* \tag{2.22}$$

Here $[\psi_i^j]^*$ denote the elements in $Lie(B_D)^*$ dual to the elements $[\psi_i^j]$. Let $pr_i : \mathfrak{g}_i^{\mathbb{C}} \to \mathfrak{b}_i$ be the $\mathcal{K}$-orthogonal projection. Then we get from the equations 2.21 and 2.22 the following expression for the moment map μ_B

$$\mu_B(\Phi_{[A]}) = (\mathcal{K}(\,\cdot\,, pr_1(Res_{p_1}\Phi_{[A]})), \ldots, \mathcal{K}(\,\cdot\,, pr_{deg(D)}(Res_{p_{deg(D)}}\Phi_{[A]}))) \, . \tag{2.23}$$

So $\Phi_{[A]}$ will satisfy the condition $\mu_B(\Phi_{[A]}) = 0$ if and only if $pr_i(Res_{p_i}\Phi_{[A]})$ is zero for every i, that is, if and only if $Res_{p_i}\Phi_{[A]} \in \mathfrak{n}_i$ for every i.

□

Finally we are going to describe the symplectic quotient of $T^*\mathcal{M}_D$ which corresponds to the quotient $\mu^{-1}(\lambda_0)/G^{\mathbb{C}}_{\lambda_0}$ of the cotangent bundle $T^*G^{\mathbb{C}}$, where $\lambda_0 \in (\mathfrak{g}^{\mathbb{C}})^*$ is a regular element. This time the quotient will not have the canonical symplectic structure of a cotangent bundle.

Let $\mathfrak{g}^{\mathbb{C}}_D \cong \bigoplus_{i=1}^{deg(d)} \mathfrak{g}^{\mathbb{C}}_i$ denote the Lie algebra of the group $G^{\mathbb{C}}_D$. Choose a regular element λ_i in each of the dual Lie algebras $(\mathfrak{g}^{\mathbb{C}}_i)^*$ and denote the regular element $(\lambda_1, \dots, \lambda_{deg(D)}) \in (\mathfrak{g}^{\mathbb{C}}_D)^*$ by λ_D.

The group $G^{\mathbb{C}}_D$ acts on the dual Lie algebra by the coadjoint action. Since each element λ_i is regular in $(\mathfrak{g}^{\mathbb{C}}_i)^*$, their stabilisers are the Cartan subgroups $H_i \in G^{\mathbb{C}}_i$. Denote the stabiliser of the element λ_D by H_D. Then clearly $H_D \cong \prod_{i=1}^{deg(D)} H_i$.

Let again the group $G^{\mathbb{C}}_D \cong \mathcal{G}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}_D$ act on the space $T^*\mathcal{M}_D$, and let as before

$$\mu_D : T^*\mathcal{M}_D \longrightarrow (\mathfrak{g}^{\mathbb{C}}_D)^*$$

be the moment map of this action. Then we can form the symplectic quotient

$$\mu_D^{-1}(\lambda_D)/H_D = (\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par} \, .$$

A question arises, whether any regular $\lambda_D \in (\mathfrak{g}^{\mathbb{C}}_D)^*$ lies in the image of the moment map

$$\mu_D : T^*_{([A],\phi)}\mathcal{M}_D \longrightarrow (\mathfrak{g}^{\mathbb{C}}_D)^*$$

restricted to the fibre above $([A], \phi)$. Recall that

$$\mu_D(\Phi_{[A]}) = (Res_{p_1}\Phi_{[A]}, \dots, Res_{p_{deg(D)}}\Phi_{[A]}) \, .$$

So, what we are asking above is the following. Given a fixed complex structure $[A]$ on the bundle adP, and fixed values $\lambda_1, \dots, \lambda_{deg(D)} \in \mathfrak{g}^{\mathbb{C}}$, does there exist a meromorphic section of the bundle $adP \otimes K$ with simple poles over the points $p_i \in D$ and with respective residues λ_i at those points.

Let

$$0 \to \mathcal{O}(adP \otimes K) \to \mathcal{O}_D(adP \otimes K) \to \mathcal{PP}_D \to 0$$

be the exact sequence of sheaves, where $\mathcal{O}(adP \otimes K)$ is the sheaf of holomorphic sections of $adP \otimes K$, $\mathcal{O}_D(adP \otimes K)$ the sheaf of meromorphic sections with simple poles in D and $\mathcal{PP}_D$ is the skyscraper sheaf of principal parts (residues in our case) of meromorphic functions over D. The long exact cohomological sequence corresponding to the above short exact sequence of sheaves has the form

$$0 \to H^0(adP \otimes K) \xrightarrow{\imath} H^0(adP \otimes K(D)) \xrightarrow{p} H^0(\mathcal{PP}) \xrightarrow{\delta} H^1(adP \otimes K) \to \dots \quad (2.24)$$

Here we again identified the meromorphic sections of $adP \otimes K$ with the holomorphic sections of $adP \otimes K(D)$ by tensoring the first ones by a section $\sigma \in H^0([D])$ which has

D as its zero divisor. So our question about the image of the moment map restricted to a specific fibre is a problem of Mittag-Leffler type. We are going to prove the following proposition

Proposition 10 *Let $G_D^{\mathbb{C}}$ act on the space $T^*\mathcal{M}_D$ in the natural way and let*

$$\mu_D : T^*_{([A],\phi)}\mathcal{M}_D \longrightarrow (\mathfrak{g}_D^{\mathbb{C}})^*$$

be the corresponding moment map. Then the element $(\lambda_1, \ldots, \lambda_{deg(D)}) \in (\mathfrak{g}_D^{\mathbb{C}})^*$ is in the image of $\mu_D/_{T^*_{([A],\phi)}\mathcal{M}_D}$ in and only if

$$\sum_{i=1}^{deg(D)} \mathcal{K}(\lambda_i, \Psi(p_i)) = 0$$

for every holomorphic section $\Psi \in H^0_{[A]}(C; adP)$.

Proof: Let the element $\widetilde{\lambda}_D \in H^0(\mathcal{PP})$ be given by

$$\widetilde{\lambda}_D = \sum_{i=1}^{deg(D)} \lambda_i \cdot \frac{dz_i}{z_i} = \sum_{i=1}^{deg(D)} \widetilde{\lambda}_i \ ,$$

where z_i is a local parameter on a neighbourhood U_i centered at $p_i \in D$. Let U_0 be an additional open set in C such that $\{U_0, U_1, \ldots, U_{deg(D)}\}$ cover C. Choose a smooth function φ_i which is equal to 1 on a disc $\Delta_{i,\epsilon}$ of radius ϵ inside U_i and zero outside U_i. Then

$$\delta(\widetilde{\lambda}_D) = \sum_{i=1}^{deg(D)} \overline{\partial}(\varphi_i \cdot \widetilde{\lambda}_i) \ .$$

So $\delta(\widetilde{\lambda}_D)$ is a global smooth $(1,1)$-form on C. Choose an element $\Psi \in H^0_{[A]}(C; adP)$. Then

$$\langle \delta(\widetilde{\lambda}_D) \ , \ \Psi \rangle = \int_C \mathcal{K}(\delta(\widetilde{\lambda}_D) \ , \ \Psi) \ .$$

By Leibnitz rule

$$\overline{\partial}\mathcal{K}(\varphi_i\widetilde{\lambda}_i \ , \ \Psi) = \mathcal{K}(\overline{\partial}(\varphi_i\widetilde{\lambda}_i) \ , \ \Psi) - \mathcal{K}(\varphi_i\lambda_i \ , \ \overline{\partial}\Psi) = \mathcal{K}(\overline{\partial}(\varphi_i\widetilde{\lambda}_i) \ , \ \Psi)$$

since $\overline{\partial}\Psi = 0$. In addition we have $\overline{\partial}\mathcal{K}(\varphi_i\widetilde{\lambda}_i) \ , \ \Psi) = d\mathcal{K}(\varphi_i\widetilde{\lambda}_i) \ , \ \Psi)$. Using Stokes' theorem we then finaly get

$$\int_C \mathcal{K}(\delta(\widetilde{\lambda}_D) \ , \ \Psi) \ = \ \int_C d\mathcal{K}(\sum_{i=1}^{deg(D)} \varphi_i\widetilde{\lambda}_i \ , \ \Psi)$$

$$= \lim_{\epsilon \to 0} \int_{C \setminus \cup U_i} d\mathcal{K}(\sum_{i=1}^{deg(D)} \varphi_i \widetilde{\lambda}_i \ , \ \Psi)$$

$$= \lim_{\epsilon \to 0} \sum_{i=1}^{deg(D)} \int_{\partial U_i} \mathcal{K}(\widetilde{\lambda}_i \ , \Psi)$$

$$= \sum_{i=1}^{deg(D)} \int_{\partial U_i} \mathcal{K}(\lambda_i \ , \ \Psi) \frac{dz_i}{z_i}$$

$$= \sum_{i=1}^{deg(D)} \mathcal{K}(\lambda_i \ , \ \Psi(p_i)) \ .$$

□

Remark 4 *In the case where the bundle P is a trivial $G^{\mathbb{C}}$-bundle over $\mathbb{CP}^1$ the constant are the only holomorphic sections of $adP \to \mathbb{CP}^1$. The above proposition then immediately gives us the description of cotangent bundle $T^*\mathcal{M}_D$ for this case. Namely for every element $\alpha \in \mathcal{M}_D$ we have*

$$T^*_\alpha \mathcal{M}_D = \{\Phi(z) = \sum_{i=1}^{deg(D)} \frac{\lambda_i}{z - p_i} \ ; \ \sum_{i=1}^{deg(D)} \lambda_i = 0\}$$

We will deal with this situation in more detail in the next chapter.

Remark 5 *In the case where the underlying holomorphic structure $[A] \in \mathcal{M}$ is stable the restriction of the moment mat on the fibres $T^*_{([A],\phi)}\mathcal{M}_D$ is surjective for every framing ϕ since $H^0_{[A]}(C; adP) = 0$ in this case. The above proposition also illustrates nicely that the non-stability of a holomorphic structure implies existence of a non-trivial automorphism group of this holomorphic structure.*

The above proposition also shows that in the case of stable $[A]$ the space

$$\mu_D^{-1}(\lambda_D) \cap T^*_{(A],\phi)}\mathcal{M}_D \subset H^0_{[A]}(C; ad \otimes K(D))$$

is an affine subspace modelled on the vector space $\imath(H^0_{[A]}(C; adP \otimes K))$ where $\imath$ is from the sequence 2.24 and it is given by $\imath(\Phi_{[A]}) = \sigma \otimes \Phi_{[A]}$ and σ is the meromorphic section of K with simple poles in the points of D.

In the following proposition we state the first part of theorem 3

Proposition 11 *Let* $((\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}, \omega_{\mathcal{M}KK})$ *denote the complex symplectic quotient space* $\mu_D^{-1}(\lambda_D)/H_D$. *Then the spaces* $(\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}$ *and* $T^*\mathcal{M}_{par}$ *are diffeomorphic, but the symplectic structures* $\omega_{\mathcal{M}KK}$ *and* ω_{can} *are different.*

Proof: The space $\mathcal{M}_D$ is a principal $G_D^{\mathbb{C}}$-bundle with the $G_D^{\mathbb{C}}$-action described at the beginning of this section. The base space of this bundle is obviously the moduli space $\mathcal{M}$ of holomorphic structures on $P \to C$. The projection map $\pi : \mathcal{M}_D \to \mathcal{M}$ is just the forgetful map which forgets the framings above the points of the divisor D. We construct Borel sub-algebra $\mathfrak{b}_{\lambda_D}$ of $\mathfrak{g}_D^{\mathbb{C}}$ corresponding to the element $\lambda_D = (\lambda_1, \cdots, \lambda_{deg(D)}) \in (\mathfrak{g}_D^{\mathbb{C}})^*$ in the obvious way. For each i let, as before $\mathfrak{h}_i \subset {\mathfrak{g}^{\mathbb{C}}}_i$ be the Cartan sub-algebra which stabilzes λ_i with respect to the coadjoint action, and let ${\mathfrak{g}^{\mathbb{C}}}_i = \mathfrak{h}_i \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}^i_\alpha \oplus \bigoplus_{\alpha \in R^-} \mathfrak{g}^i_\alpha$ be the decomposition of ${\mathfrak{g}^{\mathbb{C}}}_i$ into root spaces, the choice of positive roots being determined by the Weyl chamber containing the element λ_i. Then take $\mathfrak{b}_i$ to be $\mathfrak{b}_i = \mathfrak{h}_i \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}^i_\alpha$. Finally, let $\mathfrak{b}_{\lambda_D} = \prod_{i=1}^{deg(D)} \mathfrak{b}_i$. By $B_{\lambda_D} \subset G_D^{\mathbb{C}}$ denote the Borel subgroup corresponding to the algebra $\mathfrak{b}_{\lambda_D}$. Then B_{λ_D} acts symplecticaly and hamiltonially (as a subgroup of $G_D^{\mathbb{C}}$) on $T^*\mathcal{M}_D$. Denote the corresponding moment map by $\mu_{B_{\lambda_D}}$ and form the symplectic quotient

$$\mu^{-1}_{B_{\lambda_D}}(\lambda_D)/B_{\lambda_D} = \mathcal{M}M_{\lambda_D} \, . \tag{2.25}$$

with the induced symplectic form ω_{λ_D}. We can then define the symplectomorphism

$$\mathcal{F} : (\mathcal{M}M_{\lambda_D} \; , \; \omega_{\lambda_D}) \longrightarrow ((\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par} \; , \; \omega_{\mathcal{M}KK}) \tag{2.26}$$

precisely in the same way as we defined $\widetilde{F}$ in the formula 1.20 in section1.2. With all this at hand we see that our theorem is just a special case of proposition 4, the only difference being we are dealing here with the products $G_D^{\mathbb{C}}$ and B_{λ_D} of groups $G^{\mathbb{C}}$ and B_{λ_i} rather than with just single copies, but the change caused by this is only a notational one.

Let us only have a look at the definition of the map

$$\mathcal{R} : T^*\mathcal{M}_D \longrightarrow \mu^{-1}_{B_{\lambda_D}}(0) \, .$$

Let $\mathfrak{A}$ be a connection on the principal B_{λ_D}- bundle $\mathcal{M}_D \to \mathcal{M}_{par}$. The map $\mathcal{R}$ is defined by theformula

$$\mathcal{R}(\Phi_{[A,\phi]}) = \Phi_{[A,\phi]} - \mathfrak{A}^*_{[A,\phi]}(\mu_{B_{\lambda_D}}(\Phi_{[A,\phi]})) \, .$$

Recall that, when representing the elements $\Phi_{[A,\phi]} \in T^*_{[A,\phi]}$ as $[A]$-meromorphic sections of the bundle $adP \otimes K$, the moment map is given by

$$\mu_{B_{\lambda D}}(\Phi_{[A,\phi]}) = (Res_{p_1}\Phi_{[A,\phi]}, \ldots, Res_{p_{deg(D)}}\Phi_{[A,\phi]}) \, .$$

The mapping

$$\mathfrak{A}^*_{[A,\phi]} : (\mathfrak{b}_{\lambda_D})^* \longrightarrow T^*_{[A,\phi]}\mathcal{M}_D$$

assigns to every $(\nu_1, \ldots, \nu_{deg(D)}) \in \mathfrak{b}_{\lambda_D}$ an $[A]$-meromorphic section of $adP \otimes K$ with simple poles at the points $p_i \in D$ and having ν_i as residues there. Above we have seen that such meromorphic sections exist in the case of stable underlying holomorphic structure. The assumprion of stability is in accordance with restriction to the large open subset in $\mathcal{M}_D$ where $G_D^{\mathbb{C}}$ acts freely, since the stabiliser of $([A], \phi)$ is the automorphism group of the holomorphic structure $[A]$. We have also seen that sections with prescribed residues form an affine space, so the map $\mathfrak{A}^*_{[A,\phi]}$ chooses one particular section in this affine space. So the map $\mathcal{R}$ is well defined and our theorem is indeed a corollary of proposition 4.

It remains to show that the spaces $((\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}, \omega_{\mathcal{M}KK})$ and $(T^*\mathcal{M}_{par}, \omega_{can})$ are not symplectomorphic. Suppose for a moment that $deg(D) = 1$, i.e. that D consists of a single point. The generalisation to divisors D with $deg(D) > 0$ is immediate. The second of the above spaces is a fibre bundle

$$\begin{array}{ccc} T^*(G^{\mathbb{C}}/B) \rightarrow & T^*\mathcal{M}_{par} \\ & \downarrow \\ & \mathcal{M} \end{array}$$

while the first one is

$$\begin{array}{ccc} \mathcal{O}^{\mathbb{C}}_{\lambda_i} \rightarrow & (\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par} \\ & \downarrow \\ & \mathcal{M} \end{array} .$$

The inclusions of fibres in the above diagrams are symplectomorphisms. This means that restricting the symplectic structure ω_{can} on the fibre of the first fibration yields the usual complex canonical form on the cotangent bundle $T^*(G^{\mathbb{C}}/B)$. On the other hand restriction of the form $\omega_{\mathcal{M}KK}$ on the fibre of the second fibration is the holomorphic Kostant-Kirrilov form ω_{KK} on the complex coadjoint orbit $\mathcal{O}^{\mathbb{C}}_{\lambda_i}$. In the first chapter we have shown that $T^*(G^{\mathbb{C}}/B)$ and $\mathcal{O}^{\mathbb{C}}_{\lambda_i}$ are diffeomorphic. But the $\mathcal{O}^{\mathbb{C}}_{\lambda_i}$ equipped with the complex structure corresponding to the form ω_{KK} is a Stein manifold while the complex manifold $T^*(G^{\mathbb{C}}/B)$ has compact complex submanifolds such as the zero section $G^{\mathbb{C}}/B \subset T^*(G^{\mathbb{C}}/B)$, so the two symplectic forms are different. It is also immediately clear that ω_{can} is an exact form while $\omega_{\mathcal{M}KK}$ is not. □

2.3 Integrable systems on symplectic quotients of $T^*\mathcal{M}_D$

In subsection 2.1.3 we have seen that the dimension of the moduli space $\mathcal{M}_D$ of framed complex structures on the principal bundle $P \to C$ is given by the formula $\dim \mathcal{M}_D =$

$n(g-1)+n\cdot deg(D)$, where g is the genus of the curve C, and n is the dimension of the structure group $G^{\mathbb{C}}$. On the other hand the number of the Poisson-commuting functions $f_{i,j} : T^*\mathcal{M}_D \to \mathbb{C}$ defined in 2.6, was only $n(g-1)+\frac{1}{2}(n+r)\cdot deg(D)$, r being the rank of $G^{\mathbb{C}}$, which is less than the dimension of $\mathcal{M}_D$ and therefore does not suffice for the existence of an integrable system on $T^*\mathcal{M}_D$. As we have seen, the group $G^{\mathbb{C}}_D$ acts on the space $T^*\mathcal{M}_D$ by

$$g\cdot\Phi_{[A,\phi]} = Ad_{\psi}(\Phi_{[A,g\phi]})\ ,$$

where $\psi \in \mathcal{G}^{\mathbb{C}}$ is a representative of the element $g=[\psi]\in\mathcal{G}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}_D \cong G^{\mathbb{C}}_D$. From this and from the construction of the integrals $f_{i,j}$ it is clear that these integrals are invariant with respect to the action of $G^{\mathbb{C}}_D$ and any of its subgroups. Therefore they induce the integrals $\tilde{f_{i,j}}$ on the symplectic quotients of $T^*\mathcal{M}_D$. As shown in section1.2 these induced integrals Poisson commute, whenever the original ones do. In this section we are going to show that after descending on one of the symplectic quotients described in the previous section the dimensions of the obtained symplectic spaces and the number of the functions induced from those mentioned above coincide.

2.3.1

We begin with the cotangent bundle $T^*\mathcal{M}_{par}$. In proposition 9 we proved that $T^*\mathcal{M}_{par}\cong\mu_B^{-1}(0)/B_D$. From this we immediately get the dimension of $T^*\mathcal{M}_{par}$. Namely,

$$dimT^*\mathcal{M}_{par} = dimT^*\mathcal{M}_D - 2dimB\cdot deg(D)\ .$$

Since $dimB=\frac{1}{2}(n+r)$, this gives

$$dim\mathcal{M}_{par} = n(g-1)+\frac{1}{2}(n-r)\cdot deg(D)\ .$$

We will calculate the number of integrals $f_{i,j}$ that become trivial after passing on the quotient. In order to see what happens to $f_{i,j}$'s, we are going to split the spaces $H^0(C;K(D))^*$ in a natural way. Take the exact sequence of sheaves

$$0\to\mathcal{O}(K^d)\xrightarrow{i}\mathcal{O}_{D^d}(K^d)\xrightarrow{p}\mathcal{PP}\to 0\ , \tag{2.27}$$

where $\mathcal{O}_{D^d}(K^d)$ is the sheaf of the meromorphic sections of $K(D)$ with poles of degrees not more than d at the points of D, and $\mathcal{PP}$ is the skyscraper sheaf of the principal parts of such sections. By Serre duality we have $H^1(C\ ;\ K^d)^*\cong H^0(C\ ;\ K^{1-d})$. But by Kodaira vanishing theorem, this is equal to zero, since in our case d is always bigger than one. The sequence dual to the long cohomology sequence of 2.27 has therefore the form

$$0\to H^0(C;\mathcal{PP})^*\xrightarrow{p^*}H^0(C;K(D)^d)^*\xrightarrow{i^*}H^0(C;K^d)^*\to 0\ .$$

So p^* is an inclusion of $H^0(C;\mathcal{PP})^*$ into $H^0(C;K(D)^d)^*$, and the representatives of elements of the space $p^*(H^0(C;\mathcal{PP})^*)$ have an easy description. Think of $\varphi \in H^0(C;K(D)^d)$ as a meromorphic section of K^d with poles of degree at most d at D. Let U_i be a small neighbourhood of a point $p_i \in D$ and let z_i be a local coordinate on U_i centred at p_i. Then we can take as a basis of $p^*(H^0(C\ ;\ \mathcal{PP})^*)$ the set of forms α_i^k defined by

$$\langle \alpha_i^k\ ,\ \varphi\rangle = \int_{\gamma_i} \varphi \cdot z_i^{k-1}\ dz_i = Res_{p_i}(z_i^{k-1}\varphi(z_i))\ ,$$

where $k = 1, \ldots d$, and γ_i is a small loop around p_i lying in U_i.

Recall the equation 2.23 in the proof of proposition 9 ,

$$\mu_B(\Phi_{[A]}) = (\mathcal{K}(\ \cdot\ , pr_1(Res_{p_1}\Phi_{[A]})), \ldots, \mathcal{K}(\ \cdot\ , pr_{deg(D)}(Res_{p_{deg(D)}})))\ .$$

The map $pr : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{b}$ is the natural projection, and as we have already mentioned in proposition 9, the preimage $\mu_B^{-1}(0)$ consists of the meromorphic sections of $adP \otimes K$ with simple poles over D and with residues lying in sub-algebras $\mathfrak{n}_i \subset \mathfrak{g}_i^{\mathbb{C}}$, which are $\mathcal{K}$-orthogonal complements of the sub-algebras $\mathfrak{b}_i$. Choose a basis $\{q_l\}_{i=l,\ldots,r}$ of invariant polynomials of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$, and a basis $\{\alpha_{l,j}\}_{l=1,\ldots,r}^{j=1,\ldots,m_l}$ of the space $\bigoplus_{l=1}^r H^0(C\ ;\ K(D)^{d_l})^*$, where d_l is the degree of q_l and $m_l = (2d_l-1)(g-1) + d_l \cdot deg(D)$. Recall the definition of the integrals $f_{l,j}$ from 2.5, and 2.6:

$$f_{l,j}(\Phi_{[A,\phi]}) = \langle \alpha_{l,j}\ ,\ q_l(\Phi_{[A,\phi]})\rangle$$

In the neighbourhood U_i of the point p_i the section $q_l(\Phi_{[A,\phi]})$ can be expanded in the Laurent series of the form

$$q_l(\Phi_{[A,\phi]}) = \sum_{k=+d_l}^{1} \frac{\varphi_k}{z_i^k} + hol(z_i)\ .$$

The coefficient φ_{d_l} is the only one which depends uniquely on $Res_{p_i}\Phi_{[A,\phi]}$, and therefore $\varphi_{d_l} = q_i(Res_{p_i}\Phi_{[A,\phi]})$. Let the basis $\{\alpha_i^k\}$ of $p^*H^0(C\ ;\ \mathcal{PP})^*$ be a subset of the basis $\{\alpha_{l,j}\}_{l=1,\ldots,r}^{j=1,\ldots,m_l}$. Then, whenever $\Phi_{[A,\phi]} \in \mu_B^{-1}(0)$, we have

$$\langle \alpha_i^{d_l}\ ,\ q_i(\Phi_{[A,\phi]})\rangle = q_l(Res_{p_i}\Phi_{[A,\phi]}) = 0\ ,$$

since the sub-algebra $\mathfrak{n} \subset \mathfrak{g}^{\mathbb{C}}$ consists of nilpotent elements. Reindexing the basis $\{\alpha_{l,j}\}_{l=1,\ldots,r}^{j=1,\ldots,m_l}$ in such a way that $\alpha_i^{d_l}$ will be equal to $\alpha_{l,i}$ for every i counting the points in D and every l counting the invariant functions involved, we get

$$f_{l,i}(\Phi_{[A,\phi]}) = \langle \alpha_i^{d_l}\ ,\ q_i(\Phi_{[A,\phi]})\rangle = q_l(Res_{p_i}\Phi_{[A,\phi]})\ , \tag{2.28}$$

and therefore on $\mu_B^{-1}(0)$

$$f_{l,i}(\Phi_{[A,\phi]}) \equiv 0\,. \tag{2.29}$$

Taking the quotient by B_D we conclude:

$$\widetilde{f}_{l,i}(\Phi_{[A,\phi]}) \equiv 0 \tag{2.30}$$

on $T^*\mathcal{M}_{par}$. By an abuse of notation we denoted an element of $T^*\mathcal{M}_D$ and that of $T^*\mathcal{M}_{par}$ by the same symbol.

Next we are going to show that the relations 2.29 above are the only ones imposed on induced integrals $\widetilde{f}_{i,j}$ by the transition to the symplectic quotient $\mu_B^{-1}(0)/B_D$. Recall formula 2.17

$$f_{[\psi]}(\Phi_{[A,\phi]}) = \sum_{i=1}^{deg(D)} \mathcal{K}(\psi(p_i)\ ,\ Res_{p_i}\Phi_{[A,\phi]}) \tag{2.31}$$

for the Hamiltonian functions of the $G_D^{\mathbb{C}}$-action on $T^*\mathcal{M}_D$. These functions assign to an element $\Phi_{[A,\phi]}$ the components of its residues at the points $p_i \in D$ with respect to some choice of basis of $\bigoplus_{i=1}^{deg(D)} \mathfrak{g}^{\mathbb{C}}$. So, choose a basis $\{\xi\}_{k=1,\dots,n}^{i=1,\dots,deg(D)}$ of $\bigoplus_{i=1}^{deg(D)} \mathfrak{g}^{\mathbb{C}}{}_i$, and let the elements $\psi_k^i \in Lie(\mathcal{G}^{\mathbb{C}})$ satisfy the conditions $\psi_k^i(p_j) = \xi_k^i$. Then the equations 2.28 and 2.31 give us the following relations:

$$f_{l,i}(\Phi_{[A,\phi]}) = q_l(f_{[\psi_1^i]}(\Phi_{[A,\phi]})\ ,\dots,\ f_{[\psi_n^i]}(\Phi_{[A,\phi]}))\,. \tag{2.32}$$

In the case of B_D-action formula 2.31 changes slightly to become

$$f_{[\psi]}(\Phi_{[A,\phi]}) = \sum_{i=1}^{deg(D)} \mathcal{K}(\psi(p_i)\ ,\ pr_i(Res_{p_i}\Phi_{[A,\phi]}))\,, \tag{2.33}$$

where $pr_i : \mathfrak{g}^{\mathbb{C}}{}_i \to \mathfrak{b}_i$ are natural projections. Each factor $\mathfrak{b}_i$ in the direct sum $\mathfrak{b}_D = \bigoplus_{i=1}^{r} \mathfrak{b}_i$ is of the form $\mathfrak{b}_i = \mathfrak{h}_i \oplus \mathfrak{n}_i^+$, where $\mathfrak{h}_i \subset \mathfrak{g}^{\mathbb{C}}$ is a Cartan sub-algebra, and $\mathfrak{n}_i^+$ is the nilpotent summand of $\mathfrak{b}_i$. We have seen in section1.2, that every element ξ of a Borel sub-algebra $\mathfrak{b} \subset \mathfrak{g}^{\mathbb{C}}$ is of the form $\xi = \lambda + \alpha = Ad_b(\lambda)$ for some $\lambda \in \mathfrak{h}$, some $\alpha \in \mathfrak{n}^+$, and some b from the Borel group B corresponding to the algebra $\mathfrak{b}$. From the Ad-invariance of functions q_i, and from formulae 2.32, and 2.33 we get the expression

$$f_{l,i}(\Phi_{[A,\phi]}) = q_l(f_{[\chi_1^i]}(\Phi_{[A,\phi]})\ ,\dots,\ f_{[\chi_r^i]}(\Phi_{[A,\phi]}))\,, \tag{2.34}$$

where $\chi_j^i(p_k) = \lambda_j^i$,and $\{\lambda_j^i\}_{j=1,\dots,r}$ is a basis of the Cartan sub-algebra $\mathfrak{h}_i$. Since the number of functions $f_{l,i}$ constructed in 2.28 and the number of functions $f_{\chi_j^i}$ are the same, and since the transformation

$$(z_1,\dots,z_r) \mapsto (q_1(z_1,\dots,z_r),\dots,q_r(z_1,\dots,z_r))$$

is non-degenerate almost everywhere, formula 2.34 tells us that the systems of functions $\{f_{l,i}\}$ and $\{f_{[\chi^i_j]}\}$ are equivalent. This means that they are Hamiltonian functions corresponding to the action of the same commutative group on $T^*\mathcal{M}_D$, or in other words

$$f_{l,i} = f_{[\tau^i_1]} \tag{2.35}$$

for some different basis $\{[\tau^i_j]\}$ of the commutative sub-algebra $\bigoplus_{i=1}^r \mathfrak{h}_i \subset \mathfrak{b}_D$.

Now suppose that the restriction on the preimage $\mu_B^{-1}(0)$ induced some relation among the integrals $f_{i,j}$ in addition to those given by 2.29. Then we could construct a function $g : T^*\mathcal{M}_D \to \mathbb{C}$ commuting with all the integrals $f_{i,j}$ and being identically equal to zero on the subspace $\mu_B^{-1}(0) \subset T^*\mathcal{M}_D$. But that would mean, that g is a Hamiltonian function $f_{[\psi]}$ corresponding to the infinitesimal action of some element $[\psi] \in \mathfrak{b}_D = \bigoplus_{i=1}^r \mathfrak{b}_i \subset \mathcal{G}^{\mathbb{C}}/\mathcal{G}^{\mathbb{C}}_D$. The action of B_D on $T^*\mathcal{M}_D$ is free, therefore the homomorphism of Lie algebras

$$\mathcal{H} : \mathfrak{b}_D \longrightarrow (\mathcal{C}^\infty(T^*\mathcal{M}_D)\ ,\ \{\ ,\ \})$$

given by

$$\mathcal{H}([\psi]) = f_{[\psi]}$$

is injective, and so two Poisson-commuting Hamiltonian functions can only come from two commuting elements in $\mathfrak{b}_D$. From 2.35 then follows that the elements $[\tau^i_j]$ and $[\psi]$ of the algebra $\mathfrak{b}_D$ commute. Since $\bigoplus_{i=1}^r \mathfrak{h}_i$ is the maximal commutative sub-algebra in $\mathfrak{b}_D$, the element $[\psi]$ must lie in it, which after descending onto the quotient by B_D proves that the set of relations 2.30 is complete.

The above discussion constitutes the proof of the following proposition.

Proposition 12 *Let $f_{i,j}$ be the system of $n(g-1) + \frac{1}{2}(n+r) \cdot deg(D)$ Poisson-commuting functions on the cotangent bundle $T^*\mathcal{M}_D$. Then the number of nontrivial induced functions $\widetilde{f}_{i,j}$ on the symplectic quotient $T^*\mathcal{M}_{par} = \mu_B^{-1}(0)/B_D$ is $n(g-1) + \frac{1}{2}(n-r) \cdot deg(D)$, which is the same as the dimension of the space $T^*\mathcal{M}_{par}$, and these functions Poisson-commute. There are $r \cdot deg(D)$, r being the rank of $G^{\mathbb{C}}$, elements $f_k : T^*\mathcal{M}_D \to \mathbb{C}$ of the system $f_{i,j}$ which after passing onto the symplectic quotient yield trivial functions*

$$\widetilde{f}_k(\Phi_{[A,\phi]}) \equiv 0\,.$$

The above relations are the only ones induced on $f_{i,j}$ by taking the symplectic quotient, and therefore the functions $\widetilde{f}_{i,j}$ constitute an integrable system in the space $T^*\mathcal{M}_{par}$.

□

2.3.2

Next, we are going to prove a result analogous to the one above for the symplectic quotient $(\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par} = \mu_D^{-1}(\lambda_D)/H_D$ described in proposition 11.

Proposition 13 *Let $\widetilde{h}_{i,j} : (\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par} \to \mathbb{C}$ be the system of integrals induced from the functions $f_{i,j}$ defined on the space $T^*\mathcal{M}_D$. The number of nontrivial induced functions is $n(g-1) + \frac{1}{2}(n-r)\cdot deg(D)$, and they Poisson-commute with respect to the symplectic form $\omega_{\mathcal{M}KK}$ on $(\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}$. Let $(\lambda_1,\ldots,\lambda_{deg(D)}) = \lambda_D$. There exists a subsystem f_i^l, $i = 1,\ldots deg(D)$, $l = 1,\ldots,r$ of the set $\{f_{i,j}\}$, such that the induced functions satisfy the following relations*

$$\widetilde{h}_{l,i}(\Phi_{[A,\phi]}) \equiv q_l(\lambda_i)$$

for every index l and i. These relations are the only ones induced by the symplectic quotient while passing from $f_{i,j}$ to $\widetilde{h}_{i,j}$.

Proof: We start with the symplectic quotient $\mu^{-1}_{B_{\lambda_D}}(\lambda_D)/B_{\lambda_D} = \mathcal{M}M_{\lambda_D}$ mentioned in formula 2.25 of the proof of theorem 11. The elements in the preimage $\mu^{-1}_{B_{\lambda_D}}(\lambda_D)$ are meromorphic sections $\Phi_{[A,\phi]}$ whose residues at the points of D are of the form

$$Res_{p_i}\Phi_{[A,\phi]} = \lambda_i + \alpha$$

for some nilpotent element $\alpha \in \mathfrak{n}_i^+ \subset \mathfrak{b}_{\lambda_i}$. It is then immediately clear from the proof of the previous proposition, in particular from formula 2.28, that for every function $\widetilde{g}_{l,i}$ induced from the $f_{l,i}$ appearing in formula 2.28, we get the following identity;

$$\widetilde{g}_{l,i}(\Phi_{[A,\phi]}) \equiv q_l(\lambda_i)\,. \tag{2.36}$$

Again we used the fact that $\lambda_i + \alpha = Ad_b(\lambda_i)$ for some $b \in B_{\lambda_i}$.

The proof that besides the relations 2.36 there are no others induced on $\widetilde{g}_{i,j}$ by passing from $T^*\mathcal{M}_D$ to $\mathcal{M}M_{\lambda_D}$ is precisely the same as the proof of the analogous statement in the previous proposition. So the above proposition holds for the symplectic quotient $\mathcal{M}M_{\lambda_0}$. Recall now the diffeomorphism

$$\mathcal{F} : \mathcal{M}M_{\lambda_D} \longrightarrow (\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}$$

constructed in the proof of theorem 11. Careful inspection of the definition of the map $\widetilde{F}$ in the proof of proposition 4 shows that because of the Ad-invariance of the polynomials q_l involved in the definition of the integrals $\widetilde{g}_{i,j}$ on $\mathcal{M}_{\lambda_D}$ and $\widetilde{h}_{i,j}$ on $(\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}$, they satisfy the following relations

$$\mathcal{F}^*(\widetilde{h}_{i,j})(\Phi_{[A,\phi]}) = \widetilde{h}_{i,j}(\mathcal{F}(\Phi_{[A,\phi]})) = \widetilde{g}_{i,j}\,.$$

Again, because of the Ad-invariance of $q_l' s$, the integrals $\widetilde{g}_{l,i}$ and $\widetilde{h}_{l,i}$ assume the same constant values, so we have indeed

$$\widetilde{h}_{l,i} \equiv q_l(\lambda_i)$$

for every $l = 1, \ldots, r$ and every $i = 1, \ldots, deg(D)$. □

2.4 Spectral curve

In previous sections we have constructed systems of Poisson-commuting functions $\widetilde{h}_{i,j}$ and $\widetilde{f}_{i,j}$ on the spaces $T^*\mathcal{M}_{par}$ equipped with different symplectic structures. The number of functions in the systems $\{\widetilde{f}_{i,j}\}$ and $\{\widetilde{h}_{i,j}\}$ coincides with the dimension of $\mathcal{M}_{par}$. In this section we will prove that these systems of functions are functionally independent thus constituting integrable systems on the spaces $(T^*\mathcal{M}_{par}\ ,\ \omega_{can})$ and $((\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}\ ,\ \omega_{\mathcal{M}KK}) \cong (T^*\mathcal{M}_{par}\ , \omega_{\lambda_D})$. As shown in [Hi 1], one can encapsulate the quantities preserved under the flow of an integrable system on $T^*\mathcal{M}$ in an algebraic curve, called the spectral curve S. This curve lies in a natural way in the projectivisation of the total space of the cotangent bundle $K \to C$ and it is clear from the construction of S that the dimension of its linear system is equal to the number of the functionally independent components of Hitchin's map $\mathbf{H} : T^*\mathcal{M} \to \bigoplus_{i=1}^{r} H^0(C\ ;\ K^{d_i})$.

But there is more to the concept of spectral curve. It provides us with a fairly concrete description of the Liouville tori of the integrable system in question. Namely, these Liouville tori turn out to be Abelian varieties corresponding in a certain way to the spectral curve. In the simplest case, where the structure group of holomorphic bundles $P \to C$ is $GL(n;\mathbb{C})$ this is just the Jacobian torus of S. In the cases of other structure groups the corresponding tori are Prym varieties of S. This would eventually enable us to solve the integrable systems studied in terms of Θ-functions.

The case treated in this section is the one, where the structure group of $P \to C$ is $SL(n;\mathbb{C})$, since we just want to point out the differences stemming from the fact that we treat the spaces of parabolic holomorphic structures. These would than carry over without much difficulty to the cases of other classical structure groups following Hitchin's treatment in [Hi 1]. However, a part of discussion will be valid for the case with the arbitrary structure group.

2.4.1

We will begin with the construction of the spectral curve for the symplectic space $T^*\mathcal{M}_D$. Let $|K(D)|$ denote the total space of the line bundle $p : K(D) \to C$, and let $\widetilde{p} : p^*(K(D)) \to |K(D)|$ be the line bundle obtained as a pull-back of $|K(D)|$ by p.

The bundle $p^*(K(D))$ has an obvious tautological section w. In the local coordinates this is given by $w(z,t) = t$, where $z \in U \subset C$ is a local coordinate on the curve C, and t is the coordinate in the fibre direction. Use the same trivialization again to trivialise $p^*(K(D))_{p^{-1}(U)}$. Denote the pull-back $p^*(K(D))$ by $\mathcal{K}(\mathcal{D})$, and its tensor powers by $\mathcal{K}(\mathcal{D})^d$. Let $q_1, \ldots, q_r$ be a basis of the invariant polynomials of $\mathfrak{sl}(n;\mathbb{C})$. Then $q_i(\Phi)$, $i = 1, \ldots r$ are elements of $H^0(C\,; K(D)^{d_i})$ for every choice of $\Phi \in T^*\mathcal{M}_D$. Let $q_i^*(\Phi) \in H^0(|K(D)|\ ;\ \mathcal{K}(\mathcal{D})^{d_i})$ denote the pull-backs of $q_i(\Phi)$ by the projection p. Then we can define the section $\mathcal{Q}(\Phi)$ lying in the space $H^0(|K(D)|; \mathcal{K}(\mathcal{D})^{d_r})$ and given by the formula

$$\mathcal{Q}(\Phi) = \det\Big(p^*(\Phi(z) + Iw\Big) = w^{d_r} + \sum_{i=1}^{r} q_i^*(\Phi) \cdot w^{(d_r - d_i)}\,. \qquad (2.37)$$

Definition 7 *The spectral curve $S(\Phi)$ of an element $\Phi \in T^*\mathcal{M}_D$ is the zero locus of the section $\mathcal{Q}(\Phi)$.*

Restricting the bundle projection $\widetilde{p} : K(D) \to C$ to $S(\Phi) \subset K(D)$ gives a ramified cover

$$\hat{p} : S(\Phi) \longrightarrow C$$

the number of leaves being d_r. A point $z_0 \in C$ is a ramification point of $\hat{p}$ if and only if the element $\Phi(z_0) \in (adP \otimes K(D))(z_0) \cong \mathfrak{g}^{\mathbb{C}}$ is not regular, i.e. it lies in one or more Weyl chambers of the Cartan sub-algebra $\mathfrak{h}_{z_0}$. The choice of $\mathfrak{h}_{z_0}$ depends on the choice of the isomorphism between the fibre over z_0 and $\mathfrak{g}^{\mathbb{C}}$, but since such isomorphisms differ by adjunctions of the elements of $G^{\mathbb{C}}$, the regularity of $\Phi(z_0)$ is well-defined.

Let now

$$\Phi_t : I \longrightarrow T^*\mathcal{M}_D$$

be a path such that all the functions $f_{i,j} : T^*\mathcal{M}_D \to \mathbb{C}$ defined in 2.6 of the subsection 2.1.2 are going to be constant along it. It is then clear from the construction of functions $f_{i,j}$, that $S(\Phi_t) \equiv S(\Phi_0)$. Therefore we can talk about the spectral curve S belonging to the system of the integrals $f_{i,j}$.

We are now going to compute the dimension of the linear system S using Riemann-Roch theorem. For this purpose we have to compactify the space $|K(D)|$ by projectivizing it fibrewise. We will denote the resulting ruled complex surface $\mathbb{P}(K(D) \oplus \mathbb{C})$ by M. The curve S induces one on M in a natural way. If S is given locally as a (multi-valued) section $S(z) = y$, then the corresponding curve in M is locally the set of points with homogeneous coordinates $[y, 1]$. The resulting divisor will again be denoted by S.

Proposition 14 *The linear system $|S|$ of the divisor S in the ruled surface M has the dimension*

$$dim|S| = (d_r^2 - 1)(g-1) + (\frac{d_r^2 + d_r}{2}) \cdot deg(D) .$$

Proof: The curve S corresponds to a section $\mathcal{Q}$ of the line bundle $\mathcal{K}(\mathcal{D})^{d_r} \to |K(D)|$. Denote by $L \to M$ the line bundle obtained from $\mathcal{K}(\mathcal{D})^{d_r}$ after the projectivisation. Then we have $dim|S| = h^0(M\ ;\ L) - 1$. Riemann-Roch theorem for the surfaces has the form

$$\chi(L) = \chi(\mathcal{O}_M) + \frac{1}{2}(L \cdot L - L \cdot K_M) . \tag{2.38}$$

First we compute $\chi(\mathcal{O}_M)$. Noether's theorem gives

$$\chi(\mathcal{O}_M) = \frac{1}{12}(K_M \cdot K_M + \chi(M)) .$$

Since M is a $\mathbb{P}^1$-bundle over the curve C of genus g, we get

$$\chi(M) = \chi(\mathbb{P}^1)\chi(C) = 4(1-g) .$$

In calculating different intersection numbers we are going to make use of the fact that the second homology of a ruled surface is generated by the fibre $F \cong \mathbb{P}^1$ and the "zero section" $E \cong C$. So $K_M = \alpha E + \beta F$ for some pair of integers α and β. Using the obvious equalities $E \cdot F = 1$ and $F \cdot F = 0$, we get

$$\begin{aligned} K_M \cdot E &= \alpha E \cdot E + \beta \\ K_M \cdot F &= \alpha \end{aligned} \tag{2.39}$$

Every section of $K(D) \to C$ is homologous to E, therefore

$$E \cdot E = deg(K(D)) = (2g-2) + deg(D) .$$

From the adjunction formula

$$g = g(E) = \frac{E \cdot E + K_M \cdot E}{2} + 1$$

we then get $K_M \cdot E = -deg(D)$ and in the same way $K_M \cdot F = -2$. This gives $\alpha = -2$ and $\beta = 4(g-1) + deg(D)$ in 2.39, and so

$$K_M = -2E + (4(g-1) + deg(D))F .$$

The self-intersection number is then

$$K_M \cdot K_M = 8(1-g) .$$

Together with the Euler characteristic $\chi(M)$ this and Noether's theorem finally give

$$\chi(\mathcal{O}_M) = (1-g)\ . \tag{2.40}$$

The two remaining ingredients of Riemann-Roch formula that we have to compute are the intersection numbers $L \cdot L$ and $L \cdot K_M$. Putting $L = \gamma E + \delta F$ we get as before

$$\begin{aligned} L \cdot E &= \gamma E \cdot E + \delta F \\ L \cdot F &= \gamma \end{aligned}$$

When fixing the point $z_0 \in C$ the section $\mathcal{Q}$ restricts to a section of the line bundle $\mathcal{O}^{d_r}$ over $\mathbb{P}^1 = p^{-1}(z_0)$, so $L \cdot F = \gamma = d_r$. On the other hand the restriction of $\mathcal{Q}$ on E is a section of $K(D)^{d_r} \to C$, so $L \cdot E = d_r \cdot degK(D) = d_r(2g-2+deg(D))$. Since from this $\delta = 0$, we get $L = d_r E$ and therefore

$$L \cdot L = d_r^2(2(g-1) + deg(D))\ , \tag{2.41}$$

and

$$L \cdot K_M = -d_r \cdot deg(D)\ . \tag{2.42}$$

Putting 2.40, 2.41, and 2.42 in Riemann-Roch formula 2.38 we finally get

$$\chi(L) = (d_r^2 - 1)(g-1) + (\frac{d_r^2 + d_r}{2})deg(D)\ ,$$

which proves the proposition, since it can be seen from the Kodaira vanishing theorem that $h^1(M\ ;\ L) = 0$.

□

In the case where the structure group of the principal bundle $P \longrightarrow C$ is $SL(n;\mathbb{C})$, the value of d_r is equal to n. So , if $f_{i,j}$ is the system of Poisson-commuting functions on $(T^*\mathcal{M}_D\ ,\ \omega_{can})$ for this case, the corresponding dimension $dim|S|$ is equal to $dim(SL(n;\mathbb{C}))(g-1) + dim(B) \cdot deg(D)$, where $B \subset SL(n;\mathbb{C})$ is a Borel subgroup.

When the structure group $G^{\mathbb{C}}$ is different from $SL(n;\mathbb{C})$, the spectral curves are parametrized by some linear subsystem of the full system $|S|$ of S in M.

Next we want to see what happens to the spectral curve, after passing from $T^*\mathcal{M}_D$ to one of the symplectic quotients $T^*\mathcal{M}_{par}$ and $(\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}$ described in section2.2. First we are going to examine the second case. The treatment will be valid for the arbitrary semi-simple structure group.

Let $\Phi \in T^*_{[A,\phi]}\mathcal{M}_D$ be an arbitrary cotangent vector. Here we are going to perceive it as an A-holomorphic section of the bundle $(adP \otimes K(D))$ rather than a meromorphic section of $(adP \otimes K)$ with simple poles at D. As we have already mentioned the isomorphism between these two aspects is provided by tensoring the meromorphic section with the section $\sigma \in H^0(C\ ;[D])$. The framing ϕ then assigns to Φ a specific value $\Phi(p_i) = a_i \in \mathfrak{g}^{\mathbb{C}}$ at every point $p_i \in D$.

At a point $z_0 \in C$ the map

$$Q : \Big(adP \otimes K(D)\Big)_{z_0} \longrightarrow \mathbb{C}^r$$

defined, as before, by $Q(\Phi(z_0)) = (q_1(\Phi(z_0)), \ldots, q_{(n-1)}(\Phi(z_0)))$ sends $\Phi(z_0)$ to a point in $\mathbb{C}^r$ which represents the adjoint orbit of $\Phi(z_0)$. Let $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$ be a Cartan sub-algebra. Since the restriction on $\mathfrak{h}$ induces an isomorphism between the ring of $Ad_{G^{\mathbb{C}}}$-invariant functions on $\mathfrak{g}^{\mathbb{C}}$ and the W-invariant functions on $\mathfrak{h}$, where W is the Weyl group of $\mathfrak{g}^{\mathbb{C}}$, the adjoint orbits in $\mathfrak{g}^{\mathbb{C}}$ are parametrised by the space $\mathfrak{h}/W$. Globally the map

$$\mathbf{H} : T^*\mathcal{M}_D \longrightarrow (\mathfrak{h} \otimes K(D))/W$$

assigns to the section $\Phi \in H^0(C; adP \otimes K(D))$ a section φ in the vector bundle $(\mathfrak{h} \otimes K(D))/W$ of orbits twisted by $K(D)$. We note

$$(\mathfrak{h} \otimes K(D))/W \cong \bigoplus_{i=1}^{d_r} H^0(C\ ;\ K(D)^i)\ .$$

Recall the moment map

$$\mu_D : T^*\mathcal{M}_D \longrightarrow \bigoplus_{i=1}^{deg(D)} (\mathfrak{g}_i^{\mathbb{C}})^*$$

from proposition 7, choose a regular element $\lambda_D \in \bigoplus_{i=1}^{deg(D)} (\mathfrak{g}_i^{\mathbb{C}})^*$ and consider the preimage $\mu_D^{-1}(\lambda_D) \subset T^*\mathcal{M}_D$. For every element Φ from $\mu_D^{-1}(\lambda_D)$ the values $(\mathbf{H}(\Phi))(p_i)$ lie in the fixed orbits $\mathcal{O}_i \in ((\mathfrak{h} \otimes K(D))/W)_{p_i}$, namely those of the components λ_i of the element λ_D. In addition to that, the framing ϕ singles out a particular element $(\lambda(\Phi))_i \in \mathfrak{h} \otimes K(D)$ from its W-orbit.

The spectral curve $S(\Phi)$ intersects a fibre $K(D)_{z_0}$ in the d_r zeroes of the polynomial

$$\mathcal{Q}(w) = w^{d_r} + \sum_{i=1}^{d_r} q_i(\Phi(z_0)) \cdot w^{d_r - d_i}\ ,$$

where w is now the coordinate on the line $K(D)_{z_0}$. These d_r points (counted algebraically) are unordered and can be therefore thought of as the W-orbit in $\mathfrak{h} \otimes K(D)$ or as an element in $((\mathfrak{h} \otimes K(D))/W)_{z_0}$ corresponding to the orbit $\mathcal{O}_{\Phi(z_0)} \subset (adP \otimes K(D))_{z_0}$.

The map $\mathbf{H}$ and the polynomial $\mathcal{Q}$ are invariant with respect to the action of the group $\mathcal{G}_D^{\mathbb{C}} \subset \mathcal{G}^{\mathbb{C}}$. In addition, the elements $(\lambda(\Phi))_i$ are preserved by this action, since $g(p_i) = id$ for every $g \in \mathcal{G}_D^{\mathbb{C}}$ and every $i = 1, \ldots, deg(D)$.

Summarizing the above discussion gives the proof of the following proposition.

Proposition 15 *Let Φ be an arbitrary element in $(\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}$ and let $S(\Phi) \subset K(D)$ be its spectral curve. Then the restriction of the bundle projection*

$$\hat{p} : S(\Phi) \longrightarrow C$$

is a d_r-sheeted ramified covering map. For every $\Phi \in (\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}$ the curve $S(\Phi)$ intersects the fibres $K(D)_{p_i}$ $i = 1, \dots, deg(D)$ at the same points $\kappa_i^1, \dots, \kappa_i^{d_r}$. In addition the framing ϕ induces an assignment

$$\widetilde{\phi} : \kappa_i^j \longmapsto \lambda_i^j \ ,$$

where λ_i^j are the components of the point $(\lambda_1, \dots, \lambda_{deg(d)}) = \lambda_D$.

□

Let $|S|$ be the linear system of the spectral curves in the surface M from proposition 14. The fixing of the fibres of S above the marked points immediately gives us the following corollary of proposition 14.

Corollary 1 *The linear subsystem $|S_{\lambda_D}|$ of $|S|$ containing the spectral curves $S(\Phi)$ of the elements $\Phi \in (\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}$ has the dimension*

$$dim|S_{\lambda_D}| = (d_r^2 - 1)(g-1) + (\frac{d_r^2 - d_r}{2}) \cdot deg(D) \ .$$

□

The description of the spectral curves of the elements $\Phi \in T^*\mathcal{M}_{par}$ follows easily from what was told above.

Proposition 16 *Let Φ be an arbitrary element in the cotangent bundle $T^*\mathcal{M}_{par}$ and let $S(\Phi) \subset K(D)$ be its spectral curve. Then, as before, the map*

$$\hat{p} : S(\Phi) \longrightarrow C$$

is a d_r-sheeted ramified covering map. All the points $p_i \in D$ are ramification points of degree d_r, i.e. they are ramifications of the highest possible degree.

Proof: As before, it is enough to look at a representative of $\Phi \in T^*\mathcal{M}_{par}$ in the space $\mu_B^{-1}(0)$, and again we denote this representative by the same symbol Φ. Recalling the description of the moment map $\mu_B : T^*\mathcal{M}_D \to (\bigoplus_{i=1}^r \mathfrak{b}_i)^*$ in proposition 9, we see that for every $i = 1, \dots, deg(D)$ the cotangent $\Phi(p_i)$ lies in $\mathfrak{n} \otimes K(D)_{p_i}$ where

$\mathfrak{n}$ is some nilpotent sub-algebra of the fibre $(adP)_{p_i} \cong \mathfrak{g}^{\mathbb{C}}$. Therefore $q_j(\Phi(p_i)) = 0$ for every invariant function q_j and the defining equation for the spectral curve

$$\mathcal{Q}(w) = w^{d_r} + \sum_{i=1}^{d_r} q_i(\Phi(p_i)) \cdot w^{d_r - d_i}$$

at the point p_i collapses to $\mathcal{Q}(w) = w^{d_r}$. So the fibre $K(D)_{p_i}$ intersects the spectral curve $S(\Phi)$ at the single point $0 \in K(D)_{p_i}$.

□

2.4.2

We conclude this section by pointing out the (small) differences in the description of the Liouville tori of the integrable systems on the spaces $T^*\mathcal{M}_{par}$ and $(\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}$ compared to those on the space $T^*\mathcal{M}$. Here we will work exclusively with $G^{\mathbb{C}} = SL(n;\mathbb{C})$. Throughout this chapter we were using the adjoint representation of the structure group $G^{\mathbb{C}}$ and were associating the bundle $adP \to C$ to the principal one $P \to C$. Here we are going to use the fundamental representation of $\rho : SL(n;\mathbb{C}) \to Aut(\mathbb{C}^n)$. The associated bundle to P will then be a vector bundle $E \to C$ of rank n with a fixed determinant line bundle $Det(E) \to C$. The parabolic structures on such bundles were described in the example 2.

In the following proposition we are going to treat the case $((\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}\ ,\ \omega_{\mathcal{M}KK})$.

Proposition 17 *Let $\alpha \in \bigoplus_{i=2}^n H^0(C;K(D)^i)$ be a regular value of the map*

$$\mathbf{H} : (\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par} \longrightarrow \bigoplus_{i=2}^{n} H^0(C\ ;\ K(D)^i)\,.$$

Choose an element $\Phi \in (\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}$ such that $\mathbf{H}(\Phi) = \alpha$ and let S be the spectral curve of Φ. Denote by $\mathcal{L}$ the set of all line bundles $N \to S$ such that $Det(\hat{p}_*(N)) = Det(E)$. Then the following are true:

(a) $\mathcal{L} \subset Jac(S)^m$, where $m = d - \frac{n(n-1)}{2}(2(g-1) + deg(D))$, and $d = deg(Det(E))$.

(b) $\mathcal{L}$ is an Abelian variety.

(c) $\mathcal{L} \cong \mathbf{H}^{-1}(\alpha)$.

Proof: First we prove (a), i.e. we locate the component of $Pic(S) = Jac(S) \times \mathbb{Z}$ that will contain $\mathcal{L}$. Let $Det(E) \in Pic(C)$ be a line bundle of degree d, and let $N \to S$ be a line bundle, such that

$$Det(\hat{p}_*(N)) = Det(E)\,.$$

We claim

$$deg(N) = d - \frac{n(n-1)}{2}(2(g-1) + deg(D)) \,. \tag{2.43}$$

Indeed, let $U \subset S$ be the divisor, such that $[U] = N$ and let $\hat{p}_*(U) \subset C$ be its direct image. Then we have the equality (see e.g. [Ha])

$$Det(\hat{p}_*(N)) = Det(\hat{p}_*(\mathcal{O})) \otimes [\hat{p}_*(U)] \,. \tag{2.44}$$

The degrees of $[U]$ and $[\hat{p}_*(U)]$ are of course the same. On the other hand

$$\Big(Det(\hat{p}_*(\mathcal{O}))\Big)^2 = [-B] \,,$$

where B is the branching divisor of the map $\hat{p} : S \to C$. Using the adjunction formula and the ingredients collected in the proof of proposition 14 we can compute the genus of S:

$$\begin{aligned} g(S) &= \tfrac{1}{2}(K_M \cdot L + L \cdot L) + 1 \\ &= n^2(g-1) + 1 + \tfrac{n(n-1)}{2} \cdot deg(D) \,. \end{aligned} \tag{2.45}$$

From Riemann-Hurwitz theorem

$$g(S) = n(g(C) - 1) + 1 + \frac{1}{2} deg(B)$$

we can now extract the degree of the branching divisor, namely

$$deg(B) = n(n-1)(2(g-1) + deg(D)) \,.$$

Putting this into 2.44 we finally get the equality 2.43.

To prove (b) we note that the mapping $\hat{p}_* : U \longmapsto \hat{p}_*(U)$ depends only on the linear system of the divisor $U \subset S$ and it therefore induces a homomorphism

$$\hat{p}_* : Pic(S) \longrightarrow Pic(C) \,.$$

The set $\mathcal{L}$ is the fibre of the map

$$\widetilde{p} : [U] \longmapsto Det(\hat{p}_*([U])) \,,$$

and as we have seen in the proof of (a), this map is a composition of the homomorphism $\widetilde{p}$ and the translation

$$t : Pic(S) \longrightarrow Pic(S)$$

given by $t([V]) = [V] \otimes Det(\hat{p}_*(\mathcal{O}))$. From this we see that the fibre $\mathcal{L} = \widetilde{p}^{-1}(Det(E))$ is isomorphic to the kernel $\mathcal{K}$ of the homomorphism $\hat{p}_*$, which is an Abelian sub-variety in $Jac(S)$. The isomorphism between $\mathcal{K}$ and $\mathcal{L}$ is provided by a translation.

Finally we prove (c). Let $\Phi_A \in (\mathcal{T}^*\mathcal{M})^{\lambda_D}_{par}$ be an arbitrary element such that $\mathbf{H}(\Phi_A) = \alpha$. Then we have $S(\Phi_A) = S$. We are now using the fundamental representation of the structure group $SL(n;\mathbb{C})$, so we can express the section given in 2.37 in a more suggestive and standard way:

$$\mathcal{Q}(\Phi_A) = det(p^*(\Phi_A) + w \cdot I) \, .$$

Since w is the tautological section of the line bundle $p^*(K(D)) \to |K(D)|$, we see, that at any point $pt \in S$ of the spectral curve, $w(pt)$ is an eigenvalue of the endomorphism

$$p^*(\Phi_A)_{pt} : p^*(E)_{pt} \longrightarrow p^*(E \otimes K(D))_{pt} \, .$$

The eigenspaces are generically one-dimensional, so we get a line bundle $N \to S$ whose fibres are the eigenspaces of $p^*(\Phi_A)$. It is clear from the construction of N that $Det(\hat{p}_*(N)) = Det(E)$, so $N \in \mathcal{L}$.

Now let $N \in \mathcal{L}$ be arbitrary. By definition its direct image $\hat{p}_*(N) = E$ is a rank n holomorphic bundle over C with the prescribed determinant. The corresponding section $\Phi(N) \in H^0(C; End_0 \otimes K(D))$ is the push forward of the multiplication by the tautological section w by the projection p. We only have to see how to reproduce the parabolic structure on E from N. Recall that the parabolic structure on E at the points $p_i \in D$ is given by the choice of a flag $\mathcal{F}$

$$F_i^1 \subset F_i^2 \subset \ \ldots \ \subset F_i^n = E_{p_i} \, ,$$

which in turn is equivalent to an ordered sequence of one-dimensional linear spaces

$$E_i^1, \ldots, E_i^n,$$

provided we equipped the bundle $E \to C$ with a hermitian metric. Recall the assignment

$$\widetilde{\phi} : \kappa_i^j \longmapsto \lambda_i^j \, ,$$

from proposition 15. This gives an ordering to the fibres $N_{\kappa_i^j}$ and also to their direct images by p. So the choice of parabolic structure at the point $p_i \in D$ is given by the sequence of one-dimensional subspaces

$$\hat{p}_*(N_{\kappa_i^1}) \ , \ldots , \ \hat{p}_*(N_{\kappa_i^n})$$

of the fibre $(\hat{p}_*(N))_{p_i}$. From this we also see that the parabolic structure assigns to the cotangent $\Phi(N)$ the matrix $\widetilde{\phi}(\Phi(N))_i = diag(\lambda_i^1, \ldots, \lambda_i^n)$ at every point p_i of the divisor D.

□

Proposition 17 also holds in the case of the symplectic space $(T^*\mathcal{M}_{par} \ , \ \omega_{can})$. The only difference is in the reconstruction of the flags in the fibres E_{p_i} from the line bundles $N \to S$ over the spectral curve.

As we have seen, the marked points $p_i \in C$ are ramifications of the highest possible degree of the covering map $\hat{p} : S \to C$. As mentioned in [Hi 1] the fibre $(\hat{p}_*(N))_z$ of the push-forward bundle is given by

$$(\hat{p}_*(N))_z = \mathcal{O}_S / \mathcal{J}_{\hat{p}^{-1}(z)} ,$$

where $\mathcal{J}_{\hat{p}^{-1}(z)}$ is the ideal sheaf of $\hat{p}^{-1}(z)$. In the case, where z is a ramification point of degree n, this is equal to the jet $\mathcal{J}_{n(w)}$ of sections of N of degree n at $w = \hat{p}^{-1}(z)$. At this point we can think of elements $\varphi \in \mathcal{O}_S$ as the series $\varphi(w) = \sum_{i=1}^{\infty} a_i \, w^i$, and of the elements of $\mathcal{J}_{n(w)}$ as $\psi(w) = \sum_{i=n+1}^{\infty} b_i \, w^i$. The elements of the quotient are of the form $\chi(w) = \sum_{i=1}^{n} c_i \, w^i$ and they form the n-dimensional fibre $E_z = (\hat{p}_*(N))_z$. But the degree of w gives a natural ordering of the one-dimensional subspaces of this fibre, and thus provides us with the parabolic structure at the point z.

Chapter 3

Nahm's equations and generalized C. Neumann's problem

The theme of this chapter are Hamiltonian systems (T^*M, ω_{can}, H) which describe the motion of a particle on an arbitrary symmetric Riemannian space M under the influence of the force potential given by $V(h) = \mathcal{K}(Ad_h(\beta), \widetilde{\sigma}(\beta))$, so the Hamiltonian has the usual form $H = T + V$, where T is the kinetic energy. We think of M as being embedded in a semi-simple complex Lie group $G^{\mathbb{C}}$, so in the above expression for the potential $h \in M \subset G^{\mathbb{C}}$, and $\mathcal{K}$ is the Killing form.

The following main theorem was already announced in Introduction.

Theorem 4 *The Hamiltonian system (T^*M, ω_{can}, H) is integrable in the Liouville sense for every Riemannian symmetric space M.*

In particular we stress, that the theorem holds for the compact as well as for the non-compact Riemannian symmetric spaces. We prove this theorem by first proving its analogue for the case where $M = G^{\mathbb{C}}$. This is done in section 3.3. By imposing two involutions on $T^*G^{\mathbb{C}}$ we then obtain the proof of the above Theorem as a corollary. In section 3.2 we show that the above systems can be thought of as being related to the Hitchin's systems discussed in the previous chapter which are degenerated in a certain way. This setting provides us with the tool for proving the integrability.

In section 3.4 we describe the way in which certain integrals of the "master system" $(T^*G^{\mathbb{C}}, \omega_{can}, H)$ assume the role of constraints after passing from $T^*G^{\mathbb{C}}$ to the subspace T^*M. This process has a description in terms of confining the discriminant of the spectral curve to a certain linear subsystem.

In the last section we describe a few concrete examples of our general setup.

3.1 Nahm's equations and symmetric spaces

In this section we generalize the relation between the Nahm's equations and the motion on $\mathcal{H}^3 = SL(2;\mathbb{C})/SU(2)$, established by Donaldson in [Do], to the case of arbitrary pairs $(G^{\mathbb{C}}, \widetilde{G})$ where $G^{\mathbb{C}}$ is a semi-simple complex Lie group and $\widetilde{G}$ one of its (not necessarily compact) real forms. We then show how Nahm's equations give rise to a class of variational problems describing the motion of a particle in an arbitrary Riemannian symmetric space M under the influence of a certain force potential. More concretely, these variational problems are given by the Lagrangian

$$\mathcal{L}(h) = \int (\frac{1}{2}\|\dot{h}\|_M^2 + V_{\beta_0}(h))dt \,,$$

where $t \mapsto h(t)$ is a path in M, $\| \, \|_M$ the natural metric on M and $V_{\beta_0}(h)$ the field potential. We begin with a short description of symmetric Riemannian spaces emphasising in particular the fact that we can obtain them by imposing two different real structures on some semi-simple complex Lie group.

3.1.1

The literature discussing the symmetric spaces is vast. The main source used in this subsection is Helgason's book [He 1].

Definition 8 *A Riemannian manifold M is a globally symmetric space if every point $p \in M$ is a fixed point of an involutive isometry of M. This isometry takes any geodesic through P into itself as a curve, but reverses the parametrisation.*

Somewhat surprisingly, the presence of isometric involution is a very strict condition and reduces the vast realm of Riemannian geometry to the relatively narrow one of the homogeneous spaces, i.e. the quotients of finite dimensional Lie groups. But not even all homogeneous spaces are symmetric. The following description of symmetric spaces is due to Cartan.

Theorem 5 *Every globally symmetric Riemannian space M is a homogeneous space, i.e. there exist a real semi-simple Lie group G and a Lie subgroup U such that $M = G/U$. The metric on M is induced by the Killing metric on G. On the other hand, a homogeneous space $M = G/U$ is symmetric if and only if*

$$[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{u}, \quad [\mathfrak{u}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p},$$

where $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ is a direct sum decomposition of the Lie algebra $\mathfrak{g} = Lie(G)$ into the Lie sub-algebra $\mathfrak{u} = Lie(U)$ and a vector subspace $\mathfrak{p} \subset \mathfrak{g}$.

Proof: See e.g. [He 1]

□

Proposition 18 *Let* $\mathfrak{g}$ *be a real Lie algebra and*

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$$

its decomposition into a Lie sub-algebra $\mathfrak{u}$ *and a subspace* $\mathfrak{p}$. *Then*

$$\widetilde{\mathfrak{g}} = \mathfrak{u} \oplus i\mathfrak{p}$$

is a Lie algebra if and only if

$$[\mathfrak{u},\mathfrak{u}] \subset \mathfrak{u}, \quad [\mathfrak{u},\mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{p}. \tag{3.1}$$

Proof: It is a matter of a trivial verification. For an arbitrary pair of elements $(a+ix)$, $(b+iy) \in \widetilde{\mathfrak{g}}$. the bracket

$$[a+ix \ , b+iy] = ([a,b]-[x,y]) + i([a,y]+[x,b])$$

lies in $\widetilde{\mathfrak{g}}$ if and only if the conditions (3.1) hold. □

Definition 9 *The decomposition*

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$$

satisfying the properties 3.1 is called the Cartan decomposition of the real Lie algebra $\widetilde{\mathfrak{g}}$. *The Lie algebras* $\mathfrak{g} = \mathfrak{u} + \mathfrak{p}$ *and* $\widetilde{\mathfrak{g}} = \mathfrak{u} + i\mathfrak{p}$ *are called related with respect to the sub-algebra* $\mathfrak{u}$.

It can be shown (see [He 1]) that in the above decomposition the restriction of the Killing form $\mathcal{K}$ on $\mathfrak{u}$ is negative-definite, while the restriction on either $\mathfrak{p}$ or $i\mathfrak{p}$ is positive-definite. Hence one of the two Lie algebras $\mathfrak{g}$, $\widetilde{\mathfrak{g}}$ is always compact, since $\mathcal{K}$ is negative-definite on it. The sub-algebra $\mathfrak{u}$ is the maximal compact sub-algebra lying in the non-compact element of the pair $(\mathfrak{g},\widetilde{\mathfrak{g}})$.

Now let $G^{\mathbb{C}}$, $\widetilde{G}$ and G denote the Lie groups corresponding to the algebras $\mathfrak{g}^{\mathbb{C}}$, $\widetilde{\mathfrak{g}}$, and $\mathfrak{g}$ respectively. Here $\mathfrak{g}^{\mathbb{C}}$ is the complexification of the other two. Let

$$\tau, \widetilde{\tau} : G^{\mathbb{C}} \longrightarrow G^{\mathbb{C}}$$

be the real structures of the complex Lie group $G^{\mathbb{C}}$ having G and $\widetilde{G}$ as their real forms. Call the Lie groups G and $\widetilde{G}$ related with respect to U if their respective Lie algebras are of the form $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{p}$. This construction provides us with the following description of the symmetric spaces.

Proposition 19 *Let $M = G/U$ be a Riemannian globally symmetric space, where G is a semi-simple Lie group, and let $\widetilde{G}$ be related to G with respect to U. Then*

$$M = \{a \in \widetilde{\mathcal{H}};\ \tau(a) = a\}$$

where $\widetilde{\mathcal{H}} = G^{\mathbb{C}}/\widetilde{G}$ and τ is a real structure of $G^{\mathbb{C}}$ corresponding to the real form G. In other words, M is a simultaneous fixed point set of two involutions τ and $\widetilde{\sigma}$, where $\widetilde{\sigma}(a) = \widetilde{\tau}(a^{-1})$.

The metric

$$\langle \alpha, \beta \rangle_u = -K(u^{-1} \cdot \alpha\ ,\ u^{-1} \cdot \beta),$$

$\alpha, \beta \in T_u M$ is the natural metric on M. Being a fixed-point set of an isometry, $M \subset G$ is a totally geodesic sub-manifold of $\widetilde{\mathcal{H}}$ and of the Lie group G.

Proof: The real Lie algebras $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ have the same complexification

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g} \cong \widetilde{\mathfrak{g}} \oplus i\widetilde{\mathfrak{g}}.$$

Denote the quotient $G^{\mathbb{C}}/\widetilde{G}$ by $\widetilde{\mathcal{H}}$. The space $\widetilde{\mathcal{H}}$ can be viewed as a sub-space in $G^{\mathbb{C}}$ being the fixed point set of the involution $a \mapsto \widetilde{\tau}(a^{-1})$.

The natural metric on $\widetilde{\mathcal{H}}$ is defined by

$$\langle \alpha, \beta \rangle = -K(h^{-1} \cdot \alpha,\ h^{-1} \cdot \beta) \tag{3.2}$$

for $\alpha, \beta \in T_h\widetilde{\mathcal{H}}$. This metric is non-definite. We claim:

$$M = \{a \in \widetilde{\mathcal{H}};\ \tau(a) = a\}$$

where τ is the real structure corresponding to the real form $G \subset G^{\mathbb{C}}$. Indeed:

$$\{a \in \mathcal{H}; \tau(a) = a\} = (G^{\mathbb{C}} \cap G)/(\widetilde{G} \cap G) = G/U = M$$

It is also readily seen that the metric (3.2) restricts as a definite metric on M.

Suppose there existed a geodesic $\gamma(t)$ in $\widetilde{\mathcal{H}}$, such that $\gamma(t) \in M$ for $t_0 \leq t < t_1$ and $\gamma(t) \notin M$ for $t > t_1$. Then $\tau(\gamma) = \delta$ would be another geodesic in $\widetilde{\mathcal{H}}$ different from γ but with the property

$$\gamma(t_1) = \delta(t) \quad , \quad \dot{\gamma}(t_1) = \dot{\delta}(t_1)\,.$$

Geodesics are solutions of a second order differential equation, and once the two initial conditions are fixed, solutions of such equations are unique, which rules out the existence of the pair γ , δ. □

Suppose that the group G is compact. Then the symmetric space $M = G/U$ will also be compact. If we reverse the roles of G and $\widetilde{G}$ the resulting space $\widetilde{M}$ will be non-compact. When the groups G and $\widetilde{G}$ are related with respect to the subgroup U, the spaces M and $\widetilde{M}$ are called the dual Riemannian symmetric spaces. The elementary example of this duality is the pair $(S^2, \mathcal{H}^2)$, where S^2 is the 2-sphere, and $\mathcal{H}^2$ the hyperbolic 2-plane.

Next, we collect a few facts about the Cartan decompositions that will be needed later in the text.

Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$ (or of $\widetilde{\mathfrak{g}}$) and let $\mathfrak{u}^{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$ and $\mathfrak{p}^{\mathbb{C}} = \mathfrak{p} \oplus i\mathfrak{p}$. Then we have a direct sum decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}} .$$

Let

$$\Theta : \mathfrak{g}^{\mathbb{C}} \longrightarrow \mathfrak{g}^{\mathbb{C}}$$

be the involution of $\mathfrak{g}^{\mathbb{C}}$ having $\mathfrak{u}^{\mathbb{C}}$ as the $+1$ and $\mathfrak{p}^{\mathbb{C}}$ as the -1 eigenspace respectively. Denote by $\mathfrak{h}_{\mathfrak{p}}$ a maximal Abelian subspace in $\mathfrak{p}^{\mathbb{C}}$. By Zorn's lemma this lies in a maximal Abelian subspace $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$, i.e. in the Cartan sub-algebra $\mathfrak{h}$. Let $x \in \mathfrak{h}$ be an arbitrary element. Then $x - \Theta(x) \in \mathfrak{p}^{\mathbb{C}}$. In addition, it is easily seen that $[x - \Theta(x)\ ,\ y] = 0$ for every $y \in \mathfrak{h}$ (see [He 1], page 221), therefore $x - \Theta(x) \in \mathfrak{h}_{\mathfrak{p}}$ from which we see that $\Theta(\mathfrak{h}) \subset \mathfrak{h}$. Hence we have a direct sum decomposition

$$\mathfrak{h} = \mathfrak{h} \cap \mathfrak{u}^{\mathbb{C}} \oplus \mathfrak{h} \cap \mathfrak{p}^{\mathbb{C}} .$$

Denoting $\mathfrak{h} \cap \mathfrak{u}^{\mathbb{C}}$ by $\mathfrak{h}_{\mathfrak{u}}$, we can rewrite the above decomposition in the form

$$\mathfrak{h} = \mathfrak{h}_{\mathfrak{u}} \oplus \mathfrak{h}_{\mathfrak{p}} .$$

Let $\Delta \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}^{\mathbb{C}}$ corresponding to the Cartan sub-algebra $\mathfrak{h}$ and choose an ordering of the roots to obtain the partition $\Delta = \Delta^+ \cup \Delta^-$. Let W denote the Weyl group of $\mathfrak{g}^{\mathbb{C}}$. Recall that W is the subgroup of $GL(\mathfrak{h})$ generated by the reflections

$$s_\alpha(x) = x - 2\langle \alpha\ ,\ x\rangle \cdot \alpha^* ,$$

where $\langle \cdot\ ,\ \cdot \rangle$ denotes the duality pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$, α is a positive root, and $\alpha^* \in \mathfrak{h}$ is the unique element such that $\langle \alpha^*, \alpha \rangle = 1$ and $\langle \alpha^*\ ,\ \beta \rangle = 0$ for every $\beta \in ker(\alpha)$. Note that the hyper-surface $ker(\alpha) \subset \mathfrak{h}$ is the mirror of the reflection s_α.

Let $I^{G^{\mathbb{C}}} \subset S(\mathfrak{g}^{\mathbb{C}*})$ denote the ring of polynomials on $\mathfrak{g}^{\mathbb{C}}$ invariant with respect to the adjoint action of $G^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$ and let $I^W \subset S(\mathfrak{h}^*)$ be the ring of polynomials on $\mathfrak{h}$ invariant with respect to the action of W. Recall the well-known theorem of Chevalley:

Theorem 6 *Let the polynomial $q \in S(\mathfrak{g}^{\mathbb{C}*})$ be an element of $I^{G^{\mathbb{C}}}$. Then the restriction mapping*

$$q \mapsto q|_{\mathfrak{h}}$$

is an isomorphism between $I^{G^{\mathbb{C}}}$ and I^W. In addition the ring I^W is finitely generated, and the number of independent generators $q_1, \ldots, q_r$ is equal to the dimension of $\mathfrak{h}$.

Proof: See e.g. [Hu 2]. □

We have already mentioned, in the previous chapter, that there are many possible choices of the bases, but the set of degrees of the elements of any basis is uniquely determined by $\mathfrak{g}^{\mathbb{C}}$.

Let now $\Delta_{\mathfrak{p}}$ denote the set of all the roots $\alpha \in \Delta^+$ which do not vanish identically on $\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h}$ Obviously $\alpha \in \Delta^+ \setminus \Delta_{\mathfrak{p}}$ if and only if $\alpha|_{\mathfrak{h}_{\mathfrak{p}}} \equiv 0$.

Definition 10 *The Weyl group $W_{\mathfrak{p}}$ of the Cartan decomposition*

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$$

is the subgroup of W generated by the reflections s_α where $\alpha \in \Delta_{\mathfrak{p}}$

The group $W_{\mathfrak{p}}$ has the properties analogous to those from the second part of Chevalley's theorem.

Proposition 20 *The ring $I^{W_{\mathfrak{p}}}$ is finitely generated. The number p of independent generators equals the dimension $dim\ \mathfrak{h}_{\mathfrak{p}}$ and the degrees of generators are uniquely determined.*

Proof: See e.g. [He 2] □The number $p = dim\ \mathfrak{h}_{\mathfrak{p}}$ is called the rank of the dual symmetric spaces G/U, and $\widetilde{G}/U$ and it is equal to the dimension of the maximal totally geodesic flat sub-manifold in the respective symmetric space. This is not difficult to see. We can think of the symmetric spaces G/U and $\widetilde{G}/U$ as the subspaces $exp(Re\ \mathfrak{p}^{\mathbb{C}}) \subset G$ and $exp(Im\ \mathfrak{p}^{\mathbb{C}}) \subset \widetilde{G}$ respectively. Then the $exp(Re\ \mathfrak{h}_{\mathfrak{p}})$ and $exp(Im\ \mathfrak{h}_{\mathfrak{p}})$ are a torus and an affine space, and hence they are flat. They are totally geodesic sub-manifolds since $[\mathfrak{h}_{\mathfrak{p}}, [\mathfrak{h}_{\mathfrak{p}}, \mathfrak{h}_{\mathfrak{p}}]] \subset \mathfrak{h}_{\mathfrak{p}}$ (See [He 1]).

In section 3.4.1 the following lemma will also be needed.

Lemma 8 *Let $\mathfrak{g}^{\mathbb{C}}$ be a semi-simple Lie algebra and*

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$$

a Cartan decomposition with the corresponding involution Θ. Choose a nonzero element $x_\alpha \in \mathfrak{g}^\alpha$ in each of the root spaces $\mathfrak{g}^\alpha$. Then we have

$$\mathfrak{u}^{\mathbb{C}} = \mathfrak{h}_{\mathfrak{u}} \oplus \bigoplus_{\Delta^+ \setminus \Delta_{\mathfrak{p}}} (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}) \oplus \sum_{\alpha \in \Delta_{\mathfrak{p}}} \mathbb{C}(x_\alpha + \Theta(x_\alpha)) \quad ,$$

and

$$\mathfrak{p}^{\mathbb{C}} = \mathfrak{h}_{\mathfrak{p}} \oplus \sum_{\alpha \in \Delta_{\mathfrak{p}}} \mathbb{C}(x_\alpha - \Theta(x_\alpha)) \quad .$$

Proof: See [He 1], page 223. □

3.1.2

Let $\mathfrak{g}$ be an arbitrary semi-simple Lie algebra and let $T_i : I \to \mathfrak{g}$, $i = 1, 2, 3$ be functions from an interval to $\mathfrak{g}$. The system of equations

$$\dot{T}_i = \frac{1}{2} \sum \varepsilon_{i,j,k} \, [T_j, T_k] \quad , \; i = 1, 2, 3 \tag{3.3}$$

is called Nahm's system. These equations arise in the study of monopoles. (See e.g. [Do], [Hi 3]). There is also a more straightforward connection between Nahm's equations and the Yang-Mills theory described in [Do]. Take another function $T_0 : I \longrightarrow \mathfrak{g}$ and modify the system 3.3 slightly to get

$$\dot{T}_i = [T_0, T_i] + \frac{1}{2} \sum \varepsilon_{i,j,k} [T_j, T_k] \quad i = 1, 2, 3 \, . \tag{3.4}$$

Let $P \to \mathbb{R}^4$, $\mathbb{R}^4 = \{(t, x_1, x_2, x_3)\}$ be a trivial principal G-bundle and let $\mathfrak{g} = Lie(G)$. Define a connection A on P by

$$A = T_0 dt + T_1 dx_1 + T_2 dx_2 + T_3 dx_3.$$

Then one can check directly that the ASD equation for the connection A is equivalent to the system 3.4. The connection A is invariant with respect to the lifted translations in the directions x_1, x_2, x_3, since only the variable t is effective. The gauge transformations $u : \mathbb{R}^4 \longrightarrow G$ of the bundle P respecting this invariance are those of the form $u(t, x_1, x_2, x_3) = u(t)$. They act on the functions $T_i, i = 1, 2, 3$ by

$$\begin{array}{rcl} u(T_0) & = & Ad_u(T_0) - \dot{u}u^{-1} \\ u(T_i) & = & Ad_u(T_i) \, , \qquad i = 1, 2, 3 \end{array} \tag{3.5}$$

Since the ASD-equation is gauge invariant, the transformation 3.5 will send one solution of 3.4 into another. It is clear that the original system 3.3 is just the system 3.4 written in an appropriate gauge, namely, the one satisfying the equation

$$u^{-1}\dot{u} = T_0.$$

Denote by $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$ and define:

$$\alpha = \frac{1}{2}(T_0 + T_1) : I \longrightarrow \mathfrak{g}^{\mathbb{C}}$$

$$\beta = \frac{1}{2}(T_2 + T_3) : I \longrightarrow \mathfrak{g}^{\mathbb{C}}$$

Then one can check directly that the system 3.4 is equivalent to the pair of equations

$$\frac{d\beta}{dt} = 2[\alpha, \beta] \tag{3.6}$$

$$\frac{d(\alpha - \tau(\alpha))}{dt} = 2([\tau(\alpha), \alpha] + [\tau(\beta), \beta]) \tag{3.7}$$

Here $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is the real structure of $\mathfrak{g}^{\mathbb{C}}$ corresponding to the real form $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$.

Remark 6 *It is well known (see e.g. [D-K]), that the ASD equations for the curvature F_A of a connection A*

$$*F_A = -F_A$$

can be rewritten as a pair of equations

$$F_A^{2,0} = F_A^{0,2} = 0$$

$$(F_A, \omega) = 0$$

provided that the base space of the bundle in question is a Kähler manifold with the Kähler form ω. The first equation is the integrability condition for the complex structure on the bundle associated to A. The equations 3.6 and 3.7 are precisely this type of rewriting in our special case, invariant in three directions.

The equation 3.6 is invariant with respect to the complex gauge transformations $g : \mathbb{R} \to G^{\mathbb{C}}$ acting by

$$\begin{aligned} g(\alpha) &= Ad_g(\alpha) - \tfrac{1}{2}\dot{g}g^{-1} \\ g(\beta) &= Ad_g(\beta) \, . \end{aligned}$$

So if (α, β) is a solution of 3.6, then so is $(g(\alpha), g(\beta))$. Since $(0, \beta_0)$, $\beta_0 = const$ obviously solves (3.6), its general solution is

$$\begin{aligned} \alpha &= -\tfrac{1}{2}\dot{g}g^{-1} \\ \beta &= Ad_g(\beta_0) \end{aligned} \tag{3.8}$$

for any $g : \mathbb{R} \to G^{\mathbb{C}}$.

The key ingredient of this chapter is an interpretation of the equation 3.7 in a variational context. First for every $\alpha \in \mathfrak{g}^{\mathbb{C}}$ denote

$$\|\alpha\|^2 = \mathcal{K}(\alpha, \sigma(\alpha))$$

where $\mathcal{K}$ is the Killing form , $\sigma = -\tau$ and τ is the real structure corresponding to the real form $\mathfrak{g} \in \mathfrak{g}^{\mathbb{C}}$.

Proposition 21 (Donaldson) *Let (α, β) be a pair of functions from an interval into $\mathfrak{g}^{\mathbb{C}}$ and let $(\alpha', \beta') = (g(\alpha), g(\beta))$ denote the transformed pair for some $g : I \to G^{\mathbb{C}}$. Then the equation*

$$\frac{d(\alpha - \tau(\alpha))}{dt} = 2([\tau(\alpha), \alpha] + [\tau(\beta), \beta])$$

is the Euler-Lagrange equation for the action given by

$$\mathcal{L}(g) = \frac{1}{2} \int_0^1 (\|\alpha' + \sigma(\alpha')\|^2 + 2\|\beta'\|^2) dt. \tag{3.9}$$

Proof: Even though the proof is a straightforward generalisation of the calculation in [Do], we include it for the sake of completeness.

Let $\hat{\tau}$ denote the conjugation corresponding to the real form $G \subset G^{\mathbb{C}}$, and let the involution $\hat{\sigma} : G^{\mathbb{C}} \to G^{\mathbb{C}}$ be defined by $\hat{\sigma}(a) = \hat{\tau}(a^{-1})$. Clearly we have $d\hat{\sigma} = \sigma$. Let $\mathcal{H} \subset G^{\mathbb{C}}$ be the fixed point set of $\hat{\sigma}$, which as we know is the homogeneous space $\mathcal{H} \cong G^{\mathbb{C}}/G$. The following observations will simplify the calculation. By the group invariance we can assume $g \equiv e$, so the variation δg will take values in the tangent space $T_e G^{\mathbb{C}} \cong \mathfrak{g}^{\mathbb{C}}$. Since Nahm's equations are gauge invariant with respect to the real gauge transformations $u : I \to G$, we can further restrict the domain of the variation δg to the subspace $T\mathcal{H} \subset T_e G^{\mathbb{C}}$, where $T\mathcal{H}$ denotes the tangent space of $\mathcal{H}$ at the class $[e] \in \mathcal{H}$. Therefore $\sigma(\delta g) = \delta g$. Recall the definition of the Killing form

$$\mathcal{K}(\alpha\ ,\ \beta) = Tr(ad(\alpha)\ ,\ ad(\beta))\ .$$

The additivity of trace and the fact $Tr(ab) = Tr(ba)$ give us

$$\delta\mathcal{L}(g) = \int_0^1 Tr\begin{pmatrix} ad(\delta(\alpha+\sigma(\alpha)))\cdot ad(\alpha+\sigma(\alpha))+ \\ +2(ad(\delta\beta)\cdot ad(\sigma(\beta)) + ad(\beta)\cdot ad(\delta(\sigma\beta))) \end{pmatrix} dt\ . \tag{3.10}$$

Since $\sigma(ad(\delta\beta)\cdot ad(\sigma\beta)) = ad(\beta)\cdot ad(\delta(\sigma\beta))$, we have

$$Tr(ad(\delta\beta)\cdot(\sigma\beta)) = Tr(ad(\beta)\cdot ad(\delta(\sigma\beta)))\ .$$

Also

$$\delta\beta = \delta(Ad_g\beta) = ad_{\delta g}(\beta) = [\delta g\ ,\ \beta]\ , \tag{3.11}$$

and similarly

$$\delta\alpha = [\delta g\ ,\alpha] - \frac{1}{2}\frac{d}{dt}(\delta g)\ .$$

Hence

$$\delta(\alpha+\sigma(\alpha)) = [\delta g\ ,\ \alpha-\sigma(\alpha)] - \frac{d}{dt}(\delta g)\ . \tag{3.12}$$

Putting (3.11) and (3.12) into (3.10) we get

$$\delta\mathcal{L}(g) = \int_0^1 Tr\begin{pmatrix} \left(ad\ (\alpha+\sigma\alpha)\cdot(\frac{d}{dt}ad\ (\delta g) + ad\ [\delta g\ ,\alpha-\sigma(\alpha)])\right) \\ +2ad\ (\beta)\cdot[\sigma(\beta)\ ,\delta g] \end{pmatrix} dt\ .$$

The Jacobi identity

$$ad([\alpha,\beta]) = [ad(\alpha), ad(\beta)]$$

gives

$$Tr(ad(\alpha+\sigma(\alpha))\cdot ad[\delta g\ ,\ \alpha-\sigma(\alpha)]) = 2Tr(ad(\delta g)\cdot ad[\alpha\ ,\sigma(\alpha)])\ .$$

Integrating by parts we get

$$\int_0^1 Tr(ad(\alpha+\sigma(\alpha))\cdot \frac{d}{dt}(ad(\delta g))\ dt = -\int_0^1 Tr(\frac{d}{dt}(ad(\alpha+\sigma\alpha))\cdot ad(\delta g))\ dt\ .$$

Since $\sigma(\delta g)=\delta g$ and $\sigma([\delta g,\beta])=[\sigma(\beta),\delta g]$, we can finally write

$$\delta\mathcal{L}(g)=\int_0^1 Tr\Big(ad(\frac{d}{dt}(\alpha+\sigma(\alpha))+2[\alpha,\sigma(\alpha)]+2[\beta,\sigma(\beta)])\Big)\cdot ad\ \delta g\ dt\ .$$

Since our Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is semi-simple, $ad(a)=0$ if and only if $a=0$, and therefore the equation

$$\frac{d(\alpha+\sigma(\alpha))}{dt}+2([\alpha,\sigma(\alpha)]+[\beta,\sigma(\beta)])=0$$

is indeed the Euler-Lagrange equation for the functional $\mathcal{L}(g)$ as claimed.

□

In order to obtain the variational expression of the whole Nahm's system we put the solution 3.8 of the equation 3.6 into the action 3.9 getting:

$$\mathcal{L}(g)=\frac{1}{8}\int_0^1(\|\dot g g^{-1}+\sigma(\dot g g^{-1})\|^2+2\|Ad_g(\beta_0)\|)dt \tag{3.13}$$

Let as before $\hat\sigma(g)=\hat\tau(g^{-1})$, where $\hat\tau$ is the real structure of $G^{\mathbb{C}}$ corresponding to the real structure τ on $\mathfrak{g}^{\mathbb{C}}$. Since Nahm's system is invariant with respect to the real gauge transformations, it makes sense to try to rewrite the 3.13 in terms of $h=g\cdot\hat\sigma(g)$, $h(t)\in G^{\mathbb{C}}/G=\mathcal{H}$.

First we observe that $\dot g g^{-1}+\sigma(g^{-1}\dot g)$ lies in $T_{[e]}\mathcal{H}$ and that

$$\|\dot g g^{-1}+\sigma(g^{-1}\dot g)\|=2\|a\|$$

where a is the $T_{[e]}\mathcal{H}$ component of $\dot g g^{-1}$ with respect to the direct sum decomposition

$$\mathfrak{g}^{\mathbb{C}}=\mathfrak{g}\oplus i\mathfrak{g}=\mathfrak{g}\oplus T_{[e]}\mathcal{H}.$$

The norm $\|a\|=\mathcal{K}(a,\sigma(a))$ coincides with the Killing form when restricted to $T[e]\mathcal{H}\subset\mathfrak{g}^{\mathbb{C}}$ and it induces the natural Ad_G- invariant norm $\|\ \|_{\mathcal{H}}$ on $\mathcal{H}$. From this we get

$$\|\dot g g^{-1}+\sigma(g^{-1}\dot g)\|=\|\dot h h^{-1}\|_{\mathcal{H}}$$

It is also easy to see:

$$\|Ad_g\beta_0\|^2=\mathcal{K}(Ad_h\beta_0,\sigma\beta_0)=V_{\beta_0}(h)\ .$$

We can summarize the above discussion in the following proposition.

Proposition 22 *Let $T_i : I \longrightarrow \mathfrak{g}$ be a solution of the extended Nahm's system*

$$\dot{T}_i = [T_0, T_i] + \frac{1}{2}\sum \varepsilon_{i,j,k}[T_j, T_k] \quad i = 1, 2, 3, \tag{3.14}$$

and let $g : I \longrightarrow G^{\mathbb{C}}$ solve the equations

$$\begin{array}{rcl} \dot{g}g^{-1} & = & T_0 + iT_1 \\ Ad_g(\beta_0) & = & T_2 + iT_3 \end{array}$$

Then the path $h(t) = g(t) \cdot \hat{\sigma}(g(t)) : I \longrightarrow \mathcal{H}$ is an extremal for the action

$$\mathcal{L}(h) = \int_0^1 (\frac{1}{2}\|\dot{h}\|^2_{\mathcal{H}} + V_{\beta_0}(h))dt \tag{3.15}$$

where the potential $V_{\beta_0}(h)$ is given by $= \mathcal{K}(Ad_h\beta_0 \ , \ \sigma\beta_0)$.

Solutions of Nahm's system are naturally partitioned into orbits with respect to the action

$$\begin{array}{rcl} u(T_0) & = & Ad_u(T_0) - \dot{u}u^{-1} \\ u(T_i) & = & Ad_u(T_i) \quad i = 1, 2, 3 \end{array} \tag{3.16}$$

of the gauge group $\mathcal{U} = \{u : I \longrightarrow G\}$.

Proposition 23 *There is a one-to-one correspondence between the gauge equivalence classes of the solutions of Nahm's system and the solutions of the variational problem on $\mathcal{H}$ with the Lagrangian 3.15*

Proof: Let $h : I \longrightarrow \mathcal{H}$ be a fixed solution of our variational problem, and let $g : I \longrightarrow G^{\mathbb{C}}$ be such, that $g(t)\hat{\sigma}(g(t)) = h(t)$. Suppose $g(t)$ corresponds to a solution $T_i(t)$, $i = 0, 1, 2, 3$ of the Nahm's system in the sense of the previous proposition. It is clear that

$$\mathcal{W} = \{(g \cdot u) : I \longrightarrow G^{\mathbb{C}}; u : I \longrightarrow G\}$$

is precisely the set of all functions for which

$$(g \cdot u) \cdot \hat{\sigma}(g \cdot u) = h.$$

Now put

$$\widetilde{T}_0 + i\widetilde{T}_1 = (g \cdot u)^{-1}(g \dot{\cdot} u)$$

$$\widetilde{T}_2 + i\widetilde{T}_3 = Ad_{g \cdot u}(\beta_0)$$

From the previous proposition and from the action 3.16 then follows:

$$\widetilde{T}_i = u(T_i) \ , \quad i = 0, 1, 2, 3$$

□

As a corollary of this proposition, we see that there is a one-to-one correspondence between the solutions of the original Nahm's system 3.3, where $T_0 = 0$ and the solutions of the variational problem with the action 3.15.

Let, as in subsection 3.1.1, $M = G/U$ be a Riemannian symmetric space, $G^{\mathbb{C}}$ the complexification of the real group G,

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$$

the decomposition of the Lie algebra $\mathfrak{g} = Lie(G)$ associated to M and

$$\widetilde{\mathfrak{g}} = \mathfrak{u} \oplus i\mathfrak{p}$$

the real Lie algebra related to $\mathfrak{g}$ with respect to $\mathfrak{u}$. Let $\widetilde{G}$ be the semi-simple Lie group such that $\widetilde{\mathfrak{g}} = Lie(\widetilde{G})$. We have seen in the previous subsection that M is a totally geodesic sub-manifold in the homogeneous space $\widetilde{\mathcal{H}} = G^{\mathbb{C}}/\widetilde{G}$. We are going to conclude this section by specifying those solutions of Nahm's system 3.3 for the functions

$$T_i : I \longrightarrow \widetilde{\mathfrak{g}}$$

which correspond to the solutions of the variational problem

$$\mathcal{L}(h) = \int_0^1 (\frac{1}{2}\|\dot{h}\|^2_{\widetilde{\mathcal{H}}} + V_{\beta_0}(h))dt$$

confined to the sub-manifold M of $\widetilde{\mathcal{H}}$. When we will be looking for the non-trivial integrals of motion on a symmetric space we shall need the following propostion.

Proposition 24 *There is a one-to-one correspondence between the solutions of the variational problem on $M = G/U$ given by the Lagrangian*

$$\mathcal{L}(h) = \int_0^1 (\frac{1}{2}\|\dot{h}\|^2_{M} + V_{\beta_0}(h))dt$$

and the solutions of Nahm's system

$$\dot{T}_i + \frac{1}{2}\sum \varepsilon_{i,j,k}[T_j, T_k] = 0 \quad i = 1, 2, 3$$

such that

$$T_1, T_3 \ : I \longrightarrow \ i\mathfrak{p}$$

$$T_2 \ : I \longrightarrow \ \mathfrak{u}$$

Proof: We are going to show the following. Suppose a solution of

$$\dot{T}_i + [T_0, T_i] + \frac{1}{2}\sum \varepsilon_{i,j,k}[T_j, T_k] = 0 \quad i = 1, 2, 3$$

yields a path $h(t)$ such that for all t, $h(t) \in M$. Then there exists a gauge $u : I \to \widetilde{\mathfrak{g}}$ in which the following are true:

(i) $T_0 = 0$

(ii) $T_1, T_3, : I \longrightarrow i\mathfrak{p} \quad , \quad T_2 : I \longrightarrow \mathfrak{u}$

Recall that $h(t) = g(t) \cdot \hat{\sigma}(g(t))$, where

$$\dot{g}g^{-1} = T_0 + iT_1$$

$$Ad_g(\beta_0) = T_2 + iT_3 \ .$$

In the rest of this proof τ and $\widetilde{\tau}$ will denote the real structures of $G^{\mathbb{C}}$ corresponding to the real groups G and $\widetilde{G}$ respectively. For $h \in \widetilde{\mathcal{H}}$ to lie in M, means $\tau(h) = h$, so we get

$$\tau(g \cdot \widetilde{\tau}(g^{-1})) = (g \cdot \widetilde{\tau}(g^{-1}) \ ,$$

which is equivalent to

$$\widetilde{\tau}(g^{-1} \cdot \tau(g)) = g^{-1} \cdot \tau(g).$$

Decomposing $g \in G^{\mathbb{C}}$ uniquely into $g = b \cdot h, \ \ b \in G, \ \ h \in G^{\mathbb{C}}/G = \mathcal{H}$ we then get

$$h^2 = \widetilde{\tau}(h^2)$$

Elements in $\mathcal{H}$ have square roots, so

$$h = \widetilde{\tau}(h)$$

Factorizing h into $h = h_{i\mathfrak{u}} \cdot h_{i\mathfrak{p}}$ according to the decomposition $i\mathfrak{g} = i\mathfrak{u} \oplus i\mathfrak{p}$ gives

$$h_{i\mathfrak{u}} \cdot h_{i\mathfrak{p}} = \widetilde{\tau}(h_{i\mathfrak{u}} \cdot h_{i\mathfrak{p}}) = h_{i\mathfrak{u}}^{-1} \cdot h_{i\mathfrak{p}},$$

so $h_{i\mathfrak{u}} = e$. The above factorization is proved in [He 1] From this we conclude $g(t) \in exp(\mathfrak{u} \oplus \mathfrak{p} \oplus i\mathfrak{p})$ for every $t \in I$. We have seen in the previous proposition that a gauge transformation $u : I \to \widetilde{G}$ of the Nahm's system sends $g(t) : I \to \widetilde{G}$ into $g(t)u(t) : I \to \widetilde{G}$. Since $g(t)$ in our situation can be decomposed in the form $g(t) = a(t)c(t)$; $a(t) \in G$, $c(t) \in exp(i\mathfrak{p})$, the gauge transformation $d(t) = c(t)^{-1}$ sends $g(t)$ into a solution which takes values purely in G.

We still have some gauge freedom left, since we can decompose $u(t) = e(t)d(t)$, $e(t) \in exp(\mathfrak{u}) = U$, $d(t) \in exp(i\mathfrak{p})$. We are going to use the part $e(t)$ of the gauge to set $T_0 = 0$. This is indeed possible. Since $g(t) \in G$, we have $\alpha(t) = \frac{1}{2}\dot{g}(t)g(t)^{-1}$ and $\beta(t) = Ad_{g(t)}(\beta_0)$ taking values in $\mathfrak{g}$. On the other hand

$$\alpha(t) = \frac{1}{2}(T_0 + iT_1) \in \widetilde{\mathfrak{g}} \oplus i\widetilde{\mathfrak{g}}$$

$$\beta(t) = \frac{1}{2}(T_2 + iT_3) \in \widetilde{\mathfrak{g}} \oplus i\widetilde{\mathfrak{g}}$$

From this it follows

$$T_0, T_2 : I \longrightarrow \mathfrak{u}$$

$$T_1, T_3 : I \longrightarrow i\mathfrak{p}$$

If we choose the factor $e(t)$ of our gauge transformation so that it satisfies the equation $T_0(t) = e(t)^{-1}\dot{e}(t)$, the conditions *1.* and *2.* from the proposition are fulfilled.

□

3.2 Framed bundles over $\mathbb{CP}^1$

Here our goal will be to relate the mechanical systems given by the Lagrangian $\mathcal{L}(h)$, described in the previous section, to the Hamiltonian systems on the cotangent bundles over the moduli spaces of framed bundles which were studied in the second chapter. In order to do that we will have to modify these moduli spaces slightly, namely the divisor D of the marked points will this time consist of two points both with multiplicity two.

3.2.1

In the previous chapter we defined and studied the holomorphic principal bundles over an arbitrary Riemannian surface C with framings over the points of a divisor $D \in C$. Recall that we denoted the moduli spaces of such objects by $\mathcal{M}_D$.

Here we are going to focus on a very simple case of $\mathcal{M}_D$, namely the one where the underlying principal bundle P is the trivial $G^{\mathbb{C}}$-bundle over $\mathbb{CP}^1$. Grothendieck's theorem tells us that there is only one holomorphic trivial $G^{\mathbb{C}}$-bundle on the $\mathbb{CP}^1$, so it is clear immediately from the definition that in our case we have

$$\mathcal{M}_D \cong \Big(\prod_{i=1}^{deg(D)} G_i^{\mathbb{C}} \Big) / G^{\mathbb{C}} \,.$$

Here $G_i^{\mathbb{C}}$ is the fibre of the trivial bundle P over the marked point p_i and $G^{\mathbb{C}} = Aut(P)$ acts diagonally. Since everything is finite-dimensional, the description of the cotangent bundle $T^*\mathcal{M}_D$ will also be easier for this special case. The approach that mimics the one in the first chapter is to represent $T^*\mathcal{M}_D$ as a symplectic quotient of $T^*\Big(\prod_{i=1}^{deg(D)} G_i^{\mathbb{C}}\Big)$ with respect to the cotangent lifting of the diagonal action of $G^{\mathbb{C}}$ on $\prod_{i=1}^{deg(D)} G_i^{\mathbb{C}}$. Denote by Φ the elements of $T^*\mathcal{M}_D$ and recall that the moment map is given by.

$$\mu(\Phi) = \sum_{j=1}^{n} f_j(\Phi)\xi^j \,,$$

where $\{\xi^j\}$ is a basis of $(\mathfrak{g}^{\mathbb{C}})^*$, and f_j are appropriate Hamiltonians. Let the elements of $T^*\Big(\prod_{i=1}^{deg(D)} {G^{\mathbb{C}}}_i\Big)$ be given in the form $\Phi = \Big((g_i)_{i=1,\ldots n},\ (v_i \cdot g_i)_{i=1,..,n}\Big)$, where $g_i \in {G^{\mathbb{C}}}_i$ and $v_i \in {\mathfrak{g}^{\mathbb{C}}}_i$. The Hamiltonians f_i corresponding to the cotangent liftings of the fields

$$\widehat{\xi}_{j\,(g_i)_{i=1,\ldots,deg(D)}} = \Big((\widetilde{\xi}_j)_{g_i}\Big)_{i=1,\ldots,deg(D)} = \Big(g_i \cdot \xi_j\Big)_{i=1,\ldots,deg(D)}$$

are

$$\begin{aligned} f_j(\Phi) &= \quad \langle \alpha(\Phi)\ ,\ \widehat{\xi_j}\rangle = \textstyle\sum_{i=1}^{deg(D)} \langle v_i \cdot g_i\ ,\ g_i \cdot \xi_j\rangle \\ &= \textstyle\sum_{i=1}^{deg(D)} \langle Ad^*_{g_i}(v_i)\ ,\ \xi_j\rangle = \langle \textstyle\sum_{i=1}^{deg(D)} Ad^*_{g_i}(v_i)\ ,\ \xi_j\rangle \end{aligned} \tag{3.17}$$

Here α again denotes the tautological 1-form on the cotangent bundle. Since $\mu(\Phi) = 0$ if and only if $f_j(\Phi) = 0$ for every $j = 1, \ldots, n$ we have

$$\mu^{-1}(0) = \{((g_i), (v_i))\ ;\ \sum_{i=1}^{deg(D)} Ad^*_{g_i} v_i = 0\}\ .$$

So we finally get

$$T^*\mathcal{M}_D = \mu^{-1}(0)/G^{\mathbb{C}} = \{([g_i], (Ad^*_{g_i}(v_i));\ \sum_{i=1}^{deg(D)} Ad^*_{g_i} v_i = 0\}, \tag{3.18}$$

where $[g_i]$ denotes the $G^{\mathbb{C}}$-orbit of $(g_i)_{i=1,\ldots,n}$.

In the previous chapter we have seen that in general a fibre in the cotangent bundle $T^*\mathcal{M}_D$ is isomorphic to the space of certain holomorphic sections, namely

$$T^*_{[P]}\mathcal{M}_D = H^0(\mathbb{CP}^1\ ;\ ad\ P \otimes K(D))\ , \tag{3.19}$$

where $[P] \in \mathcal{M}_D = (\prod_{i=1}^{deg(D)} G^{\mathbb{C}}_i)/G^{\mathbb{C}}$. Since in the case of $\mathbb{CP}^1$ we have $K = \mathcal{O}(-2)$, we see from tha above that the elements $\Phi \in T^*\mathcal{M}_D$ are polynomials of degree $deg(D) - 2$ with the coefficients from the Lie algebra $\mathfrak{g}^{\mathbb{C}}$. After trivialising the bundles $T^*G^{\mathbb{C}}_i$, the elements $\Phi \in T^*\mathcal{M}_D$ are determined by $deg(D)$ values $v_i \in \mathfrak{g}^{\mathbb{C}}_i$ such that their sum is zero. The Legendre interpolation formula provides us with the appropriate polynomial

$$\Phi(z) = \sum_{i=1}^{deg(D)} \Phi(p_i) \cdot L_i(z)\ , \tag{3.20}$$

where $L_i(z) = (\prod_{j\neq i}(p_j - p_i))^{-1} \prod_{j\neq i}(z - p_i)$. Denote $c_i = (\prod_{j\neq i}(p_j - p_i))^{-1}$. A polynomial of the form (3.20) is in general of degree $deg(D) - 1$. Since in our case it is of degree $deg(D) - 2$, we get a condition

$$\sum_{i=1}^{deg(D)} c_i \Phi(p_i) = 0\ ,$$

which is just a rewriting of the one in (3.18). So the comparison of the expressions (3.18) and (3.19) gives

$$Ad^*_{g_i}(v_i) = c_i \Phi(p_i) \; , \quad i = 1, ..., deg(D) \; .$$

The multiplication by $s = \left(\prod_{i=1}^{deg(D)} (z - p_i) \right)^{-1}$, as it was already mentioned in the previous chapter, induces the isomorphism between $H^0(\mathbb{CP}^1 \; ; \; K(D))$ and the space of meromorphic functions on $\mathbb{CP}^1$ with degree one poles at the marked points $p_i \in D$. Using this isomorphism, we can write

$$\widehat{\Phi}(z) = \sum_{i=1}^{deg(D)} \frac{c_i \Phi(p_i)}{(z - p_i)} = \sum_{i=1}^{deg(D)} \frac{Ad^*_{g_i}(v_i)}{(z - p_i)} \tag{3.21}$$

3.2.2

Next we will introduce the degenerated framed bundles over $\mathbb{CP}^1$. Since the construction is infinitesimal it makes sense for bundles over arbitrary Riemann surfaces, therefore we shall write C instead of $\mathbb{CP}^1$ in the definition and the proposition bellow.

Let p_1 and p_2 be two points in D. A framing of a bundle $P \to C$ at p can be thought of as a 0-jet of a trivialisation of P at $p \in C$. When the points p_1 and p_2 coalesce, two 0-jets give rise to a 1-jet of a trivialisation of $P \to C$ at $p_1 = p_2$. If we let coalesce a set of $(k+1)$ points the framings at those points will degenerate into one k-jet of a trivialisation at the point where the $(k+1)$ points gether.

Definition 11 *Let $P \to C$ be a principal bundle and $D \in C$ a positive divisor with elements whose degrees may exceed 1. A degenerated framing on $P \to C$ over D is a choice of k_i-jets of trivialisations at the points $p_i \in supp(D)$ and where $k_i = deg(p_i) - 1$. A k-jet of a trivialisation is called a k-framing of $P \to C$ at a point p of degree $k+1$.*

Here we will deal only with degenerated framed bundles with 1-framings.

Proposition 25 *Let $P \to C$ be a principal $G^{\mathbb{C}}$-bundle and let $p \in C$ be a point of degree two. Then the 1-framings at p are parametrised by $TG^{\mathbb{C}}$.*

Proof: Let φ_1 and φ_2 be two 1-framings of $\pi : P \to C$ at p, and let

$$\widetilde{\varphi_1} \; , \; \widetilde{\varphi_2} : P_{/U} \longrightarrow U \times G^{\mathbb{C}}$$

be two local trivialisations such that their 1-jets are φ_1 and φ_2 respectively. Denote by $pt \in P_p$ the point for which $\varphi_1(pt) = (\pi(pt) \; , \; e) \in U \times G^{\mathbb{C}}$. Then $\varphi_2(pt) = (\pi(pt) \; , \; g)$

for some element $g \in G^{\mathbb{C}}$, so the 0-jet parts of 1-framings ϕ_1 and ϕ_2 differ by an element $g \in G^{\mathbb{C}}$. It remains to compare the first derivatives. For $a \in P_{/U}$ and $i = 1, 2$ we have

$$\varphi_i(a) = (\pi(a)\ ,\ \rho_i(a))\ ,$$

where $\rho_i : P_{/U} \to G^{\mathbb{C}}$ is a $G^{\mathbb{C}}$-equivariant map. Clearly it is enough to compare the derivatives $d\rho_1$ and $d\rho_2$, and because of $G^{\mathbb{C}}$-equivariance we can restrict ourselves to the linear maps $(d\rho_1)_{pt}$ and $(d\rho_2)_{pt}$. The target spaces of these maps are different, so we will look at $dL_g \circ (d\rho_1)_{pt} = (\widetilde{d\rho_1})_{pt}$. Abusing the notation, we will write simply $(d\rho_1)_{pt}$ instead of $(\widetilde{d\rho_1})_{pt}$.

Let $v_P \in T_{pt}P$ and $v_g \in T_g G^{\mathbb{C}}$ denote the vectors $\frac{d}{dt}|_{t=0}(exp(tv) \cdot pt)$ and $d(L_g)v$ respectively, where $v \in \mathfrak{g}^{\mathbb{C}}$. Then

$$(d\rho_1)_{pt}(v_P) = (d\rho_2)_{pt}(v_P) = v_g \tag{3.22}$$

for every $v \in \mathfrak{g}^{\mathbb{C}}$. Every vector $u \in T_{pt}P$ can be expressed as $u = (v_1(u))_P + c_1(u)k_1$ and $u = (v_2(u))_P + c_2(u)k_2$, where $k_1 \in \ker(d\rho_1)_{pt}$ and $k_2 \in \ker(d\rho_2)_{pt}$ and $c_i(u)$ are constants. Once we fix k_1 and k_2 these expressions of u are unique. From 3.22 we get

$$(d\rho_i)_{pt}(u) = (v_i(u))_g \quad i = 1, 2\ .$$

We will show that the difference $((d\rho_1)_{pt} - (d\rho_2)_{pt})(u)$ is a scalar multiple of an element $v_{(k_2)} \in T_g G^{\mathbb{C}}$, and this element is independent of $u \in T_{pt}P$.

We express k_2 in the form $k_2 = (v(k_2))_P + \alpha k_1$. Then

$$\begin{aligned} u &= (v_2(u))_P + c_2(u)k_2 \\ &= (v_2(u))_P + c_2(u)(v(k_2))_P + \alpha k_1) \end{aligned}$$

So $(v_2(u))_P - (v_1(u))_P = c_2(u)(v(k_2))_P$, and therefore

$$((d\rho_1)_{pt} - (d\rho_2)_{pt})(u) = c_2(u)(v(k_2))_g\ .$$

So we can conclude that the maps $d\rho_1$, $d\rho_2$ and therefore 1-framings φ_1 and φ_1, φ_2 differ by an element $(g\ ,\ v(k_2)_g) \in T_g G^{\mathbb{C}} \subset TG^{\mathbb{C}}$. □

Remark 7 *Above considerations can be recast in a different terminology. Let φ be a 1-framing of $P \to C$ at $p \in C$ and let $\rho : P_{/U} \to G^{\mathbb{C}}$ be as above. Then $\ker(d\rho)_{pt} \subset T_{pt}P$is a $G^{\mathbb{C}}$-equivariant distribution of horizontal subspaces in TP along the fibre P_p. There is a uniquely defined 1-form $\Omega = \rho^{-1}d\rho$ on P_p such that $\ker(d\rho)_{pt} = \ker\Omega_{pt}$. This form can be thought of as a 0-jet of a flat connection on P at the point $p \in C$. A k-framing at p would then correspond to a choice of a usual framing at p together with a choice of a $(k-1)$-jet of a flat connection at p.*

Now we return to the moduli spaces of framed structures on $\mathbb{CP}^1$. We will only be interested in the case where the divisor D^d consists of two double points. From proposition 25 and the theorem of Grothendieck we immediately see that the moduli space of holomorphic structures on the trivial $G^{\mathbb{C}}$-bundle $P \to \mathbb{CP}^1$ with 1-framings over the divisor D is equal to

$$\mathcal{M}_{D^d} \cong (TG^{\mathbb{C}} \times TG^{\mathbb{C}})/G^{\mathbb{C}} .$$

Trivializing from the left gives us $\mathcal{M}_{D^d} \cong ((G^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}) \times (G^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}))/G^{\mathbb{C}}$. Since $G^{\mathbb{C}}$ acts diagonally on $(G^{\mathbb{C}} \times G^{\mathbb{C}})$ and trivially on the copies of $\mathfrak{g}^{\mathbb{C}}$, we finally get

$$\mathcal{M}_{D^d} \cong \mathfrak{g}^{\mathbb{C}} \times \ (G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}} \ \times \mathfrak{g}^{\mathbb{C}} .$$

We will obtain the form of the elements in the space $T^*\mathcal{M}_{D^d}$ easily from 3.21. First put $p_2 = p_1 + \epsilon$ and define

$$\widetilde{\Phi}(p) = \Big((p-p_3)(p-p_4)\Big)^{-1} \cdot \Phi(p)$$

Then a calculation gives

$$\lim_{p_2 \to p_1} \Big(\frac{c_1\Phi(p_1)}{z-p_1} + \frac{c_2\Phi(p_2)}{z-p_2}\Big) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\frac{\widetilde{\Phi}(p+\epsilon)}{z-(p+\epsilon)} - \frac{\widetilde{\Phi}(p)}{z-p}\right)\Bigg|_{p=p_1} = \frac{\widetilde{\Phi}(p_1)}{(z-p_1)^2} + \frac{(\frac{d}{dp}\widetilde{\Phi})(p_1)}{(z-p_1)}$$

We proceed in the same way to calculate the limit $p_3 \to p_4$. Renaming

$$\widetilde{\Phi}_1(p) = (p-p_4)^{-2}\Phi(p), \quad \widetilde{\Phi}_2(p) = (p-p_1)^{-2}\Phi(p)$$

we then finally get the following expression for the elements $\widehat{\Phi} \in T^*\mathcal{M}_{D^d}$:

$$\widehat{\Phi}(z) = \frac{\widetilde{\Phi}_1(p_1)}{(z-p_1)^2} + \frac{(\frac{d}{dp}\widetilde{\Phi}_1)(p_1)}{(z-p_1)} + \frac{(\frac{d}{dp}\widetilde{\Phi}_2)(p_4)}{(z-p_4)} + \frac{\widetilde{\Phi}_2(p_4)}{(z-p_4)^2}$$

To get the element $\Phi \in T^*_{[P]}\mathcal{M}_{D^d} = H^0(\mathbb{CP}^1;\ ad(P) \otimes K(D))$ we have to multiply $\widehat{\Phi}$ by $r = (z-p_1)^2(z-p_4)^2$. Since the coefficient at z^3 of the resulting polynomial must be zero, we get :

$$(\frac{d}{dp}\widetilde{\Phi}_1)(p_1) = -(\frac{d}{dp}\widetilde{\Phi}_2)(p_4).$$

So the condition $\sum_{i=1}^4 \Phi(p_1) = 0$ degenerates into the condition $(\frac{d}{dp}\widetilde{\Phi}_1)(p_1) = -(\frac{d}{dp}\widetilde{\Phi}_2)(p_4)$, while the other two terms in the expression of Φ are arbitrary. Summarizing what was told above gives us the proof of the following proposition.

Proposition 26 *Let $T^*\mathcal{M}_D$ be the cotangent bundle of the moduli space of framed $G^{\mathbb{C}}$- principal bundles over $\mathbb{CP}^1$, and let $deg(D) = 4$. If we divide the four points of D into two pairs and let the points in the pairs coalesce, then the degenerate cotangent bundle will be of the form*

$$T^*\mathcal{M}_{D^d} \cong T^*\Big(TG^{\mathbb{C}} \times TG^{\mathbb{C}}\Big)/G^{\mathbb{C}} \cong T^*\Big(\mathfrak{g}^{\mathbb{C}} \times (G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}\Big).$$

The elements of this cotangent bundle can be represented in the form

$$\widehat{\Phi}(z) = \frac{\widetilde{\Phi}_1(p_1)}{(z-p_1)^2} + \frac{(\frac{d}{dp}\widetilde{\Phi}_1)(p_1)}{(z-p_1)} + \frac{(\frac{d}{dp}\widetilde{\Phi}_2)(p_4)}{(z-p_4)} + \frac{\widetilde{\Phi}_2(p_4)}{(z-p_4)^2},$$

where $(\frac{d}{dp}\widetilde{\Phi}_1)(p_1) = -(\frac{d}{dp}\widetilde{\Phi}_2)(p_4)$. More explicitly, the elements in $T^*_{(\alpha_1,[g_1,g_2],\alpha_2)}\mathcal{M}_{D^d}$ can be written in the form

$$\widehat{\Phi}(z) = \frac{Ad^*_{g_1}(\alpha_1)}{(z-p_1)^2} + \frac{Ad^*_{g_1}(v)}{(z-p_1)} + \frac{Ad^*_{g_2}(u)}{(z-p_4)} + \frac{Ad^*_{g_2}(\alpha_2)}{(z-p_4)^2},$$

where $Ad^*_{g_1}(v) = -Ad^*_{g_2}(u)$.

□

3.2.3

In this subsection we are treating the Nahm's system

$$\dot{T}_i + \frac{1}{2}\sum \varepsilon_{i,j,k}[T_j, T_k] = 0 \quad i = 1,2,3 \tag{3.23}$$

in which the maps T_i take values in the complex semi-simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$. We will interpret this system in two ways, exhibiting that it can yield motions on two different spaces: on the moduli space $\mathcal{M}_{D^d}$ on the one hand and on the homogeneous space $(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}$ on the other.

Motion on $\mathcal{M}_{D^d}$

Define new functions

$$\begin{aligned} \alpha &= (T_2 + iT_3) \quad : I \longrightarrow \mathfrak{g}^{\mathbb{C}} \\ \gamma &= (T_2 - iT_3) \quad : I \longrightarrow \mathfrak{g}^{\mathbb{C}} \\ \beta &= -2iT_1 \quad : I \longrightarrow \mathfrak{g}^{\mathbb{C}} \end{aligned} \tag{3.24}$$

The system 3.23 is then equivalent to the system

$$\begin{aligned} \dot\alpha &= \tfrac{1}{2}[\beta,\alpha] \\ \dot\gamma &= \tfrac{1}{2}[-\beta,\gamma] \\ \dot\beta &= [\gamma,\alpha] \\ -\dot\beta &= [\alpha,\gamma] \end{aligned} \tag{3.25}$$

(The reason for this redundant writing will become apparent below.)

The equation 3.25 can be interpreted as an equation for a path

$$\widehat{\Phi}_t = \frac{\alpha(t)}{(z-p_1)^2} + \frac{\beta(t)}{(z-p_1)} + \frac{-\beta(t)}{(z-p_2)} + \frac{\gamma(t)}{(z-p_2)^2} \,:\, I \longrightarrow T^*\mathcal{M}_{D^d}.$$

The system (3.25) can be expressed more economically. Represent the element $\widehat{\Phi}$ as a polynomial, i.e. multiply it by $r = (z-p_1)^2(z-p_2)^2$ and then send the point p_1 into 0 and the point p_2 into $\infty \in \mathbb{CP}^1$. (All the configurations of two points on $\mathbb{CP}^1$ are equivalent under the Möbius transformations .) The element $\widehat{\Phi}$ will then assume the form

$$\Phi = \alpha + z\beta + z^2\gamma \,,$$

and the system 3.25 is then equivalent to the Lax equation

$$\dot\Phi_t = \frac{1}{2}\Big[\frac{d}{dz}(\Phi)_t \;,\; \Phi_t\Big] \,. \tag{3.26}$$

Motion on $(G^{\mathbb{C}} \times \overline{G^{\mathbb{C}}})/G^{\mathbb{C}}_r$

The system (3.25) can be given essentially the same variational interpretation as the Nahm's system. The complexification of $\mathfrak{g}^{\mathbb{C}}$ is $\mathfrak{g}^{\mathbb{C}} \times \overline{\mathfrak{g}^{\mathbb{C}}}$, likewise $(G^{\mathbb{C}})^{\mathbb{C}} = G^{\mathbb{C}} \times \overline{G^{\mathbb{C}}}$. The subgroup

$$G^{\mathbb{C}}_r = \{(g,g); g \in G^{\mathbb{C}}\} \subset (G^{\mathbb{C}} \times \overline{G^{\mathbb{C}}})$$

is the real form, corresponding to the real structure $\tau(g_1,g_2) = (g_2,g_1)$. The homogeneous space $(G^{\mathbb{C}} \times \overline{G^{\mathbb{C}}})/G^{\mathbb{C}}_r$ consists of classes represented by elements of the form (g, g^{-1}). On the Lie algebra level the real form is $\mathfrak{g}^{\mathbb{C}}_r = \{(v,v);\ v \in \mathfrak{g}^{\mathbb{C}}\}$ and its "imaginary" complement $j{\mathfrak{g}^{\mathbb{C}}}_r = \{(v,-v);\ v \in \mathfrak{g}^{\mathbb{C}}\}$. Denoting

$$\begin{aligned} \mathcal{A} &= (\alpha,\gamma) &&= (T_2,T_2) + (iT_3,-iT_3) &&: I \longrightarrow (\mathfrak{g}^{\mathbb{C}})^{\mathbb{C}} \\ \mathcal{B} &= (\beta,-\beta) &&= (-2T_1, 2T_1) &&: I \longrightarrow (\mathfrak{g}^{\mathbb{C}})^{\mathbb{C}} \end{aligned}$$

the equations 3.25 become

$$\begin{aligned} \dot{\mathcal{A}} &= [\mathcal{B}, \mathcal{A}] \\ \dot{\mathcal{B}} &= [\tau(\mathcal{A}), \mathcal{A}] \,. \end{aligned} \tag{3.27}$$

The equations 3.27 are analogous to the equations (3.6) and (3.7) from subsection 3.1.2, the only difference being that the system above is written in the gauge corresponding to the one where $T_0 = 0$ for the usual form of Nahm's equations. From the first equation in (3.27) we get

$$\begin{aligned} \mathcal{A} &= Ad_{(g_1,g_2)}(\alpha_0, \gamma_0) \\ \mathcal{B} &= (\dot{g}_1 g_1^{-1}, \dot{g}_2 g_2^{-1}) = (\dot{g}_1 g_1^{-1}, -\dot{g}_1 g_1^{-1}) \,. \end{aligned}$$

Then, as we have seen in 3.1.2

$$h(t) = (g_1, g_2) \cdot \tau(g_1, g_2)^{-1} = \left(g_1 g_2^{-1}, (g_1 g_2^{-1})^{-1}\right) : I \longrightarrow (G^{\mathbb{C}} \times \overline{G^{\mathbb{C}}})/G_r^{\mathbb{C}} \,,$$

satisfying the second equation in 3.27 is a solution of the variational problem

$$\mathcal{L}(h) = \int_0^1 \Big(\frac{1}{2}\Big\|\dot{h}\Big\|^2_{(G^{\mathbb{C}}\times\overline{G^{\mathbb{C}}})/G_r^{\mathbb{C}}} + V_{\beta_0}(h)\Big)dt \tag{3.28}$$

on the homogeneous space $(G^{\mathbb{C}} \times \overline{G^{\mathbb{C}}})/G_r^{\mathbb{C}}$, which is isomorphic to the complex group $G^{\mathbb{C}}$. The metric in the above action is given by the scalar product

$$\langle (a, -a), (b, -b) \rangle = \widetilde{\mathcal{K}}((a, -a), (b - b)) = \mathcal{K}(a, -b) + \mathcal{K}(-b, a) = -2\mathcal{K}(a, b)$$

where $\mathcal{K}$ is the Killing form on the group $G^{\mathbb{C}}$, and the potential $V_{\beta_0}(h)$ by

$$\begin{gathered} V_{\beta_0}(h) = V_{(\alpha_1,\alpha_2)}((g_1 g_2^{-1}, g_2 g_1^{-1})) = \\ \widetilde{\mathcal{K}}(Ad_h(\beta_0), -\tau(\beta_0)) = \widetilde{\mathcal{K}}((Ad_{g_1 g_2^{-1}}(\alpha_1), Ad_{g_2 g_1^{-1}}(\alpha_2)) \;,\; (-\alpha_2, -\alpha_1)) \,. \end{gathered}$$

We now bring the two motions described together. In order to provide the common ground for the two constructions, we have to turn the Lagrangian

$$\mathcal{L}(h) = \int_0^1 \Big(\frac{1}{2}\Big\|\dot{h}\Big\|^2_{(G^{\mathbb{C}}\times\overline{G^{\mathbb{C}}})/G_r^{\mathbb{C}}} + V_{\beta_0}(h)\Big)dt$$

in an appropriate Hamiltonian using the Legendre transformation. But in our case the phase space in the Hamiltonian setting is just the cotangent bundle of the configuration space, and the force potential is dependent uniquely on the position, so the Legendre transformation is of the simplest and the most usual kind, giving the Hamiltonian

$$H = \frac{1}{2}\|\dot{h}\|^2_{(G^{\mathbb{C}}\times\overline{G^{\mathbb{C}}})/G_r^{\mathbb{C}}} - V_{\beta_0}(h) \quad ,$$

which is the total energy of our moving particle.

In addition, we replace the usual complexification $(G^{\mathbb{C}})^{\mathbb{C}} = G^{\mathbb{C}} \times \overline{G^{\mathbb{C}}}$ by the product $G^{\mathbb{C}} \times G^{\mathbb{C}}$. The involution $\tau(g_1, g_2) = (g_2, g_1)$ will then not be a real structure anymore but a holomorphic involution. Nevertheless it is obvious that all the above considerations can be rewritten in terms of $G^{\mathbb{C}} \times G^{\mathbb{C}}$ rather than $G^{\mathbb{C}} \times \overline{G^{\mathbb{C}}}$. The equations 3.27 then represent the motion on $G^{\mathbb{C}} \times G^{\mathbb{C}}$ given by the Hamiltonian

$$H = \frac{1}{2}\|\dot{h}\|^2_{(G^{\mathbb{C}}\times G^{\mathbb{C}})/G^{\mathbb{C}}_r} - V_{\beta_0}(h) \ .$$

Recall proposition 26, where $T^*\mathcal{M}_{D^d}$ was represented in the form

$$T^*\mathcal{M}_{D^d} \cong T^*\Big((G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}_r) \times (\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}})\Big) \ .$$

Denote by $[g_1, g_2]$ the elements of $(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}_r$ and by (α_1, α_2) those of $(\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}})$. Using these coordinates we can write

$$\widehat{\Phi}_{([g_1,g_2],(\alpha_1,\alpha_2))} \in T^*_{([g_1,g_2],(\alpha_1,\alpha_2))}\mathcal{M}_{D^d} \ .$$

Define the square $[g_1, g_2]^2 \in (G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}_r)$ to be the class containing the element $(g_1, g_2) \cdot \tau(g_1, g_2)^{-1} = (g_1 g_2^{-1}, g_2 g_1^{-1})$. Let

$$\pi : T^*\mathcal{M}_{D^d} \longrightarrow T^*\Big((G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}{}_r\Big)$$

be the obvious projection. Then the straightforward inspection of the above two interpretations of the Nahm's system gives the proof of the following proposition.

Proposition 27 *Let $(T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}_r, \omega_{can}, H)$ be the Hamiltonian system corresponding to the variational problem*

$$\mathcal{L}(h) = \int_0^1 (\frac{1}{2}\left\|\dot{h}\right\|_{(G^{\mathbb{C}}\times G^{\mathbb{C}})/G^{\mathbb{C}}_r} + V_{\beta_0}(h))dt$$

via the Legendre transformation. Then the path

$$2\pi(\widehat{\Phi}_{([g_1(t),g_2(t)]^2 \ , \ (\alpha_1,\alpha_2))}) : I \longrightarrow T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}_r$$

is a solution of this system if and only if $\widehat{\Phi}_t : I \to T^*\mathcal{M}_{D^d}$ is a solution of the system (3.25).

□

3.3 Integrability

We are going to exploit the embedding established above of the systems given by the Lagrangians $\mathcal{L}(h)$ into the cotangent bundle $T^*\mathcal{M}_{D^d}$. We have seen in the second chapter that the spaces $T^*\mathcal{M}_D$ "contain" natural integrable Hamiltonian systems. It will turn out that the systems given by $\mathcal{L}(h)$ are closely related to them.

3.3.1

In subsection 3.2.3 we have expressed the Nahm's equations in many ways. Here we are going to use the rewriting in the form of the Lax equation, which arises when the marked points p_1 and p_4 assume the antipodal values 0 and ∞ in $\mathbb{CP}^1$.

$$\dot{\Phi}_t(z) = \Big[(\frac{d}{dz}\Phi_t)(z)\ ,\ \Phi_t(z) \Big]\ . \tag{3.29}$$

Fixing an arbitrary point $z_o \in \mathbb{CP}^1$, the Lax equation tells us that at every $t \in I$ the vector $\dot{\Phi}_t(z_0)$ lies in the tangent space of the adjoint orbit $\mathcal{O}_{\Phi(z_0)} \subset \mathfrak{g}^{\mathbb{C}}$ at the point $\Phi(z_0)$ of that orbit. So the solutions of the equation (3.29) are of the form

$$\Phi_t(z) = Ad_{g(t)}(\Phi_0(z))$$

for some path $g(t) : I \to G^{\mathbb{C}}$. Therefore, for any $Ad_{G^{\mathbb{C}}}$-invariant function q the value $q(\Phi_t(z))$ will depend only on the initial condition $\Phi_0(z)$, and we can proceed in the same way as in the previous chapter.

Choose a basis $\{q_1, \ldots, q_r\}$ of the invariant polynomials of $\mathfrak{g}^{\mathbb{C}}$ whose respective degrees are d_i, and define the map

$$\mathbf{H} : T^*\mathcal{M}_{D^d} \longrightarrow \bigoplus_{i=1}^{r} H^0(\mathbb{CP}^1; K(D)^{d_i}) \tag{3.30}$$

by the formula $\mathbf{H}(\Phi) = (q_1(\Phi), \ldots, q_r(\Phi))$. Clearly the mapping $\mathbf{H}$ is constant along the solutions of the equation (3.29). Taking into account that $deg(D) = 4$ and $deg(K) = -2$ we get

$$\bigoplus_{i=1}^{r} H^0(\mathbb{CP}^1;\ K(D)^{d_i}) = \bigoplus_{i=1}^{r} H^0(\mathbb{CP}^1;\ \mathcal{O}(2)^{d_i}) = \bigoplus_{i=1}^{r} H^0(\mathbb{CP}^1;\ \mathcal{O}(2d_i))\ ,$$

so the dimension of the above vector space is $d = \sum_{i=1}^{r}(2d_i + 1) = dim(\mathfrak{g}^{\mathbb{C}}) + 2r$.

Let the marked points assume again a generic position , and suppose, that $\widehat{\Phi}_t$ solves the first two equations of (3.25). Such cotangents lie in the subspace of $T^*\mathcal{M}_{D^d}$ consisting of the elements of the form

$$\widehat{\Phi}(z) = \frac{Ad^*_{g_1}(\alpha_1)}{(z-p_1)^2} + \frac{\dot{g}_1 g_1^{-1}}{(z-p_1)} + \frac{\dot{g}_2 g_2^{-1}}{(z-p_2)} + \frac{Ad^*_{g_2}(\alpha_2)}{(z-p_2)^2}$$

for constant α_1, α_2. This allows us to define the mapping $\widetilde{\mathbf{H}}$ on the subspace $T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}}$ of $T^*\mathcal{M}_{D^d}$ by:

$$\widetilde{\mathbf{H}}([g_1, g_2], (\dot{g}_1 g_1^{-1}, \dot{g}_2 g_2^{-1})) = \mathbf{H}(\widehat{\Phi}(z)). \tag{3.31}$$

This mapping is well defined. Any representative of the class $([g_1, g_2], (\dot{g}_1 g_1^{-1}, \dot{g}_2 g_2^{-1}))$ is of the form

$$\Big([gg_1, gg_2], (Ad_g(\dot{g}_1 g_1^{-1}), Ad_g(\dot{g}_2 g_2^{-1})\Big) .$$

Because of the Ad-invariance of $\mathbf{H}$, the mapping $\widetilde{\mathbf{H}}$ is independent of the choice of the representative.

Choose a basis $\{e^i\}$ of the dual space $(\bigoplus_{i=1}^r H^0(\mathbb{CP}^1;\ \mathcal{O}(2d_i)))^*$ and define the functions

$$H_i : T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}} \longrightarrow \mathbb{C}$$

by the formula

$$H_i([g_1, g_2]\ ,\ (\dot{g}_1 g_1^{-1}, \dot{g}_2 g_2^{-1})) = \langle \widetilde{\mathbf{H}}(\widehat{\Phi})\ ,\ e^i \rangle\ . \tag{3.32}$$

These functions are constant along the solutions of (3.29). Elements of the spaces $H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))$ from the direct sum $\bigoplus_{i=1}^r H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))$ are polynomials of degree $2d_i$, so one natural way of choosing the components of the mapping $\widetilde{\mathbf{H}}$ is to take the coefficients of these polynomials. Let again $p_1 = 0$ and $p_2 = \infty$. After multiplying by $r = (z - p_1)^2 (z - p_2)^2$, $\widehat{\Phi}$ becomes $\Phi(z) = Ad_{g_1}(\alpha_1) + z(\dot{g}_1 g_1^{-1}) + z^2 Ad_{g_2}(\alpha_2)$ and $\widetilde{\mathbf{H}}(\Phi) = (q_1(\Phi), \ldots, q_r(\Phi))$ can be expanded into polynomials

$$q_i(\Phi(z)) = \sum_{j=0}^{2d_i} H_{j+\sum_{k=1}^{i-1}(2d_k+1)} \cdot z^j\ . \tag{3.33}$$

Then we quickly see that $2r$ components of the mapping $\widetilde{\mathbf{H}}$ are trivial, since they are constant functions on $T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}}$. Let q_i be an invariant polynomial. Denote by $q_{i,0}$ the constant term of $q_i(\Phi(z))$. We have

$$q_{i,0} = q_i(\Phi(0)) = q_i((Ad_{g_1}(\alpha_1)) = q_i(\alpha_1)\ .$$

Denote now by $q_{i,\infty}$ the highest term of the polynomial $q_i(\Phi(z)$. Here we get

$$q_{i,\infty} = q_i(lim_{z\to\infty}(1/z^2)\Phi(z)) = q_i((Ad_{g_2}(\alpha_2)) = q_i(\alpha_2).$$

In terms of 3.36 we have

$$H_{\sum_{k=1}^{i-1} 2d_k}(\Phi) \equiv q_i(\alpha_1)$$

and

$$H_{2d_i+\sum_{k=1}^{i-1}(2d_k+1)}(\Phi) \equiv q_i(\alpha_2)\ .$$

So the number of non-trivial (non-constant) functions H_i on $T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}}$ is $\sum_{i=1}^r (2d_i + 1) - 2r = n = \dim G^{\mathbb{C}}$. In the sequel we will reindex the system $\{H_i\}$ so that the first n functions will be the non-trivial ones.

Theorem 7 *Let $T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}}$ be the cotangent bundle over the homogeneous space $(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}} \cong G^{\mathbb{C}}$, let ω_{can} be the natural symplectic form on this cotangent bundle, and let the Hamiltonian function H be given by*

$$H = \frac{1}{2}\|(2\dot{g}_1 g_1^{-1}, 2\dot{g}_2 g_2^{-1})\|^2_{(G^{\mathbb{C}}\times G^{\mathbb{C}})/G_r^{\mathbb{C}}} - V_{(\alpha_1,\alpha_2)}(g_1 g_2^{-1}, g_2 g_1^{-1}) ,$$

where $V_{(\alpha_1,\alpha_2)}(g_1 g_2^{-1}, g_2 g_1^{-1}) = \widetilde{\mathcal{K}}(Ad_{(g_1 g_2^{-1}, g_2 g_1^{-1})}(\alpha_1, \alpha_2) \ , \ (-\alpha_2, -\alpha_1))$. Then the Hamiltonian system

$$(T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}} \ , \ \omega_{can} \ , \ H\Big)$$

is integrable in the Liouville sense. The functions H_i defined in (3.32) are a set of n Poisson-commuting functionally independent integrals of our system.

Proof: First we show that the Hamiltonian H is contained in the set of integrals H_i, $i = 1, \ldots, n$. The Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is semi-simple, so there will be no linear Ad-invariant polynomial. For the quadratic invariant polynomial we can simply take $q_2(a) = \mathcal{K}(a, a)$, where $\mathcal{K}$ is the Killing form. Keeping in mind that $\dot{g}_1 g_1^{-1} = \dot{g}_2 g_2^{-1}$, we get for the z^2 coefficient of $K(\Phi, \Phi)$ the following expression.

$$K(\dot{g}_1 g_1^{-1}, -\dot{g}_1 g_1^{-1}) + 2K(Ad_{g_1}\alpha_1, Ad_{g_2}\alpha_2) = -K(\dot{g}_1 g_1^{-1}, \dot{g}_1 g_1^{-1}) + 2K(\alpha_1, Ad_{g_1^{-1}g_2}\alpha_2) =$$

$$\frac{1}{2}\|(2\dot{g}_1 g_1^{-1}, 2\dot{g}_2 g_2^{-1})\|^2 + \widetilde{K}(Ad_{(g_1 g_2^{-1}, g_2 g_1^{-1})}(\alpha_1, \alpha_2) \ , \ (-\alpha_2, -\alpha_1))$$

This is precisely the Hamiltonian H of our problem.

Next we are going to show that these integrals Poisson-commute. Let $\mathbf{H}$ be as defined in (3.30). From its definition (3.32) we see that every H_i is a restriction of a function

$$\langle \mathbf{H} \ , \ e^i \rangle : T^*\mathcal{M}_{D^d} \longrightarrow \mathbb{C} ,$$

to the subspace where the coordinates (α_1, α_2) are fixed. We are going to define an extension $\widehat{H}_i$ of H_i on the whole space $T^*\mathcal{M}_{D^d}$ in the following manner. The space $T^*\mathcal{M}_{D^d}$ can be split into two factors:

$$T^*\mathcal{M}_{D^d} \cong (T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}}) \times (T^*(\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}})). \tag{3.34}$$

With respect to this factorization and after trivializing, the elements of $T^*\mathcal{M}_{D^d}$ can be represented in the form

$$\Big((\ (t_1, t_2) \ , \ ([g_1, g_2])) \ , \ ((t_3, t_4) \ , \ (A_1, A_2) \ ) \Big) .$$

Abbreviate the above expression by writing it as $(t_1, t_2, t_3, t_4, [g_1, g_2], A_1, A_2)$.

Let then $\widehat{H}_i$ be given by the formula

$$\widehat{H}_i(t_1, t_2, t_3, t_4, [g_1, g_2], A_1, A_2) \stackrel{def}{=} \langle \mathbf{H} \ , \ e^i \rangle (t_1, t_2, Ad^*_{g_1}(\alpha_1), Ad^*_{g_2}(\alpha_2), [g_1, g_2], \alpha_1, \alpha_2)$$

$$= H_i([g_1, g_2], (t_1, t_2)) \,,$$

where α_1 and α_2 are fixed. It is easily seen that the functions $\widehat{H}_i$ do Poisson-commute on $T^*\mathcal{M}_{D^d}$. As we have seen, $T^*\mathcal{M}_{D^d}$ is the symplectic quotient of the space $T^*(TG^{\mathbb{C}} \times TG^{\mathbb{C}})$ with respect to the action of $G^{\mathbb{C}}$. Trivializing $T^*(TG^{\mathbb{C}} \times TG^{\mathbb{C}})$ we get $\bigoplus_{i=1}^4 (\mathfrak{g}^{\mathbb{C}}{}_i)^* \oplus (\mathfrak{g}^{\mathbb{C}} \times G^{\mathbb{C}} \times G^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}})$. The functions $\widehat{H}_i$ descend from the $Ad_{G^{\mathbb{C}}}$-invariant functions F_i on $T^*(TG^{\mathbb{C}} \times TG^{\mathbb{C}})$. In the trivialization the functions F_i depend only on the variables in $(\oplus_{i=1}^4 \mathfrak{g}_i^{\mathbb{C}})^*$, and therefore they Poisson-commute. The induced functions $\widehat{H}_i$ on $T^*\mathcal{M}_{D^d}$ then also commute.

Denote the symplectic form on the first factor of the space (3.34) by ω_{can} and the one on the second factor by ω_1. Then with respect to the (3.34) we obviously have $\omega = \omega_{can} + \omega_1$, where ω is the symplectic form on $T^*\mathcal{M}_{D^d}$. Of course, the same is true for the Poisson brackets. So we have

$$\{\widehat{H}_i, \widehat{H}_j\} = \{\widehat{H}_i, \widehat{H}_j\}_{can} + \{\widehat{H}_i, \widehat{H}_j\}_1 = 0.$$

In our case the coordinates (α_1, α_2) are fixed, so we have $\{\widehat{H}_i, \widehat{H}_j\}_1 = 0$, and therefore finally

$$\{\widehat{H}_i, \widehat{H}_j\}_{can} = \{H_i, H_j\}_{can} = 0 \,,$$

as claimed.

It remains to prove the functional independence of our integrals H_i i.e. of the components of the mapping

$$\widetilde{\mathbf{H}} : T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}} \longrightarrow \bigoplus_{i=1}^{r} H^0(\mathbb{CP}^1; \mathcal{O}(2d_i) \ .$$

The most suitable and geometrically suggestive way of doing this is by use of spectral curve. For the case where $G^{\mathbb{C}} = SL(n : \mathbb{C})$, this is done in proposition 29. However, this approach runs into difficulties when $G^{\mathbb{C}}$ is not a classical group. Here we give a different proof, based on the approach of Miscenko and Fomenko in [Mi-Fo], which works for arbitrary semisimple $G^{\mathbb{C}}$.

Functional independece of integrals $\{H_i\}$ means that generically $dH_1 \wedge \ldots \wedge dH_n \neq 0$, which in turn is equivalent to the map

$$d\widetilde{\mathbf{H}}_\Phi : T_\Phi(T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}}) \longrightarrow T_{\widetilde{\mathbf{H}}(\Phi)}(\bigoplus_{i=1}^{r} H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))) \cong \bigoplus_{i=1}^{r} H^0(\mathbb{CP}^1; \mathcal{O}(2d_i)),$$

having rank n for a generic Φ. We will show that already the restriction of $d\widetilde{\mathbf{H}}_\Phi$ to the vertical subspace $T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}} \subset T_\Phi(T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}})$ has rank n for a generic choice of Φ .

In [Mi-Fo] the autors study Poisson-commuting functions on complex coadjoint orbits $\mathcal{O}^{\mathbb{C}} \subset (\mathfrak{g}^{\mathbb{C}})^*$. Let $x \in \mathcal{O}^{\mathbb{C}}$ and let $\alpha \in (\mathfrak{g}^{\mathbb{C}})^*$ be a fixed element and let

$$q_i(x + z\alpha) = \sum_{j=1}^{d_i} f_\alpha^{i,j}(x) \cdot z^j \ ,$$

where q_i is an Ad^*-invariant function and z is an indeterminate. Denote by $df_\alpha^{i,j}$ the derivative of $f_\alpha^{i,j}$ with respect to x. While proving theorem 4.2 of [Mi-Fo] the authors establish the following fact.

Lemma 9 (Miscenko, Fomenko) *Let* $b = (\dim\mathfrak{g}^{\mathbb{C}} + \mathrm{rank}\mathfrak{g}^{\mathbb{C}})/2$. *Then 1-forms* $df_\alpha^{i,j}$, *for* $i = 1, \dots, r$ *and* $j = 0, \dots, d_i - 1$ *span a* b*-dimensional subspace of* $\mathfrak{g}^{\mathbb{C}}$ *for a generic choice of* x *and* α.

Recall that our system of integrals $\{H_i\}$ is obtaianed by expanding the functions

$$q_i(\Phi(z)) = q_i(\alpha + z\beta + z^2\gamma)$$

with respect to z. Clearly we get the same (reindexed) system if we expand

$$q_i(\widetilde{\Phi}(z)) = q_i(\frac{1}{z}\alpha + \beta + z\gamma) = \sum_{i=-d_i}^{d_i} H^{i,j}(\alpha, \beta, \gamma) \ .$$

Let $dH^{i,j}$ denote the derivatives of $H^{i,j}(\alpha, \beta, \gamma)$ with respect to β. We will use lemma 9 to prove that $dH^{i,j}$ span $(\mathfrak{g}^{\mathbb{C}})^*$. This is clearly equivalent to the map

$$d\widetilde{\mathbf{H}} : T^*_\Phi(G^{\mathbb{C}} \times G^{\mathbb{C}})G^{\mathbb{C}}_r \longrightarrow \bigoplus_{i=1} rH^0(\mathbb{CP}^1; \mathcal{O}(2d_i)$$

having rank $n = \dim\mathfrak{g}^{\mathbb{C}}$.

Let now w be another indeterminate and let

$$\Psi(z, w) = \frac{1}{w}\alpha + \beta + z\gamma \ .$$

Then $\Phi(z) = \Psi(z, z)$. For every Ad^*-invariant q_i we have

$$q_i(\Psi(z, w)) = \sum_{j=0}^{d_i} f_i^j(w) \cdot z^j$$

and

$$q_i(\Psi(z, w)) = \sum_{i=0}^{d-i} g_i^j(z) \cdot w^{-j}$$

and further

$$q_i(\Psi(z,w)) = \sum_{j=0}^{d_i}\Big(\sum_{k=0}^{d_i-j} h^j_{i,k}\cdot w^{-k}\Big)\cdot z^j \tag{3.35}$$
$$q_i(\Psi(z,w)) = \sum_{j=0}^{d_i}\Big(\sum_{k=0}^{d_i-j} h^k_{i,j}\cdot z^k\Big) w^{-j}\ . \tag{3.36}$$

Since $\Psi(z,\infty) = \beta + z\gamma$ and $\Psi(0,w) = \beta + \frac{1}{w}\alpha$, lemma 9 tells us that the forms $dh^0_{i,j}$ span a b-dimensional subspace $E_{(\beta,\gamma)} \subset (\mathfrak{g}^{\mathbb{C}})^*$ and the forms $dh^j_{i,0}$ span another b-dimensional subspace $F_{(\beta,\alpha)}$. For a generic choice of α,β,γ the spaces $E_{(\beta,\gamma)}$ and $F_{(\beta,\alpha)}$ intersect transversally. The r-dimensional intersection is spanned by $dh^0_{i,0}$ From $q_i(\Psi(z,z)) = q_i(\Phi(z))$ and from 3.36 we get

$$dH^{i,j} = \sum_{k=0}^{d_i-j} dh^{k-j}_{i,j} \quad \text{for} \quad j \le 0 \tag{3.37}$$
$$dH^{i,j} = \sum_{k=0}^{d_i-j} dh^{k}_{i,k+j} \quad \text{for} \quad j > 0 \tag{3.38}$$

where $q_i(\Phi(z)) = \sum_{j=-d_i}^{d_i} H^{i,j}\cdot z^j$.

Let τ be the real structure of $\mathfrak{g}^{\mathbb{C}}$, corresponding to the compact real form, and let, as usual, $\mathcal{K}$ denote the Killing form on $\mathfrak{g}^{\mathbb{C}}$. Then $\alpha \to \mathcal{K}(\alpha,\tau(\alpha))$ defines a norm $\|\cdot\|$ on $\mathfrak{g}^{\mathbb{C}}$ and it also induces one on $(\mathfrak{g}^{\mathbb{C}})^*$. The forms $dh^k_{i,j}$ are polynomial functions of $(\alpha,\beta,\gamma) \in ((\mathfrak{g}^{\mathbb{C}})^*)^3$. More precisely, components of α occur in $dh^k_{i,j}$ with the power j, those of γ with the power k, and the components of β have the power $d_i - j - k$. Therefore in each sum in 3.38 the first summand has the highest degree in β. (In all other summands β occurs with degree at least two less than in the first one.) From this we see

$$\lim_{\|\beta\|\to\infty} \frac{dH^{i,j}}{\|dH^{i,j}\|} = \frac{dh^j_{i,0}}{\|dh^j_{i,0}\|} \quad \text{for} \quad j \le 0$$
$$\lim_{\|\beta\|\to\infty} \frac{dH^{i,j}}{\|dH^{i,j}\|} = \frac{dh^0_{i,j}}{\|dh^0_{i,j}\|} \quad \text{for} \quad j > 0\,.$$

Therefore for a large enough β the forms $dH^{i,j}$ span the same space as the forms $\{dh^j_{i,0}\ ,\ dh^0_{i,j}\}$, that is the whole $(\mathfrak{g}^{\mathbb{C}})^*$ for a generic choice of (α,β,γ). Choose a basis in $(\mathfrak{g}^{\mathbb{C}})^*$ and n 1-forms $dH^{i,j}$ that span $(\mathfrak{g}^{\mathbb{C}})^*$. Compose an $n \times n$ matrix having $dH^{i,j}$ as columns and denote its determinant by $F(\alpha,\beta,\gamma)$. Then F is a polynomial function of (α,β,γ) which is different from zero on an open set in $((\mathfrak{g}^{\mathbb{C}})^*)^3$ as we have seen above. Therefore it is different from zero for a generic choice of (α,β,γ). □

3.4 Constraints

We have seen in proposition 24 that the functions in Nahm's equations describing the motion on a symmetric space G/U, are subject to the conditions:

$$\begin{array}{rcl} T_1, T_3 : & I \longrightarrow & i\mathfrak{p} \\ T_2 : & I \longrightarrow & \mathfrak{u} \end{array} \quad . \tag{3.39}$$

Previously in the text we have shown the existence of n Poisson-commuting integrals of motion for the Hamiltonian system $(T^*G^{\mathbb{C}}\ ,\ \omega_{can}\ ,\ H)$. When we reduce the system to the symplectic sub-manifold $T^*(G/U)$, the number of the independent integrals should decrease by the right amount. More precisely, some of the integrals should become trivial, i.e. constant on the whole subspace $T^*(G/U)$, turning into the constraints of the system. Of course, there is no reason to expect that some subset of the integrals H_i defined by 3.32 should provide such constraints, but certain functions $F_j(H_1, \ldots, H_n)$ will. In this subsection we are going to construct the appropriate number of independent constraints $F_j(H_1, \ldots, H_n)$. Obviously, the number of constraints will be equal to the dimension *dim U* of the subgroup U.

At the end we describe the spectral curve S of our system. The constraints obtained in subsections 3.4.1 and 3.4.2 have a natural description in terms of the linear system $|S|$ of the curve S.

3.4.1

At the begining we are going to relax the condition 3.39 somewhat to demand

$$\begin{array}{rcl} T_1, T_3 : & I \longrightarrow & \mathfrak{p}^{\mathbb{C}} \\ T_2 : & I \longrightarrow & \mathfrak{u}^{\mathbb{C}} \end{array} \quad , \tag{3.40}$$

corresponding to the complexified direct sum decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$$

of the Lie algebra into a sub-algebra and the tangent space of a symmetric space at the point corresponding to $id \in G^{\mathbb{C}}$.

Recall that the section $\Phi \in H^0(\mathbb{CP}^1; \mathfrak{g}^{\mathbb{C}} \otimes \mathcal{O}(2))$ satisfying the Lax equation 3.26 is of the form

$$\Phi(z) = (T_2 + iT_3) - z(2iT_1) + z^2(T_2 - iT_3)\ ,$$

which can be rewritten as

$$\Phi(z) = (1 + z^2)T_2 + i(T_3 - z2T_i - z^2T_3)\ . \tag{3.41}$$

So, if the functions $T_i, i = 1, 2, 3$ satisfy the conditions 3.40, we have

$$\Phi(z_0) \in \mathfrak{p}^{\mathbb{C}}$$

for $z_0 = \pm i$, and this circumstance will be the source of the sought for constraints $F_j(H_1, \ldots H_n)$.

Let $\mathfrak{h}_{\mathfrak{p}}$ denote the maximal Abelian subspace in $\mathfrak{p}^{\mathbb{C}}$ and let $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$ be the Cartan sub-algebra of the form

$$\mathfrak{h} = \mathfrak{h}_{\mathfrak{u}} \oplus \mathfrak{h}_{\mathfrak{p}} \ ,$$

introduced in subsection 3.1.1. Let $\Phi \in \mathfrak{g}^{\mathbb{C}}$ be an arbitrary element and $\mathcal{O}_\Phi$ its adjoint orbit. Then, as is well known, the intersection $\mathcal{O}_\Phi \cap \mathfrak{h}$ is non-empty, or more precisely,

$$\mathcal{O}_\Phi \cap \mathfrak{h} = W \cdot x \ ,$$

where $W \cdot x$ denotes the orbit of an element $x \in \mathcal{O}_\Phi \cap \mathfrak{h}$ with respect to the Weyl group action on $\mathfrak{h}$. In the sequel we will need the following proposition which describes a special occurrence of the above mentioned general situation.

Lemma 10 *Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$ be a Cartan decomposition of $\mathfrak{g}^{\mathbb{C}}$ and let Φ_p lie in the subspace $\mathfrak{p}^{\mathbb{C}}$. Then the intersection $\mathcal{O}_{\Phi_p} \cap \mathfrak{h}$ lies in the subspace $\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h}$, or more precisely*

$$\mathcal{O}_{\Phi_p} \cap \mathfrak{h} = W_{\mathfrak{p}} \cdot x \ ,$$

where $W_{\mathfrak{p}}$ is the subgroup of W defined in definition 10, and $x \in \mathcal{O}_{\Phi_p} \cap \mathfrak{h}$ is arbitrary.

Proof: First we show that the intersection $\mathcal{O}_{\Phi_p} \cap \mathfrak{h}_{\mathfrak{p}}$ is non-empty. We are going to prove the statement for the compact real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$, but that will imply the validity of the complex case. By an abuse of notation, we will denote the intersection $\mathfrak{h}_{\mathfrak{p}} \cap \mathfrak{g}$ simply by $\mathfrak{h}_{\mathfrak{p}}$ in this proof.

Let $\Phi_p \in \mathfrak{p} \subset \mathfrak{g}$. Take an element $y \in \mathfrak{h}_{\mathfrak{p}}$, such that its centraliser within $\mathfrak{p}$ is $\mathfrak{h}_{\mathfrak{p}}$. Define the function

$$f : \mathcal{O}^U_{\Phi_p} \longrightarrow \mathbb{R}$$

on the Ad_U-orbit of Φ_p by

$$f(Ad_u \Phi_p) = \mathcal{K}(Ad_u \Phi_p \ , \ y) \ .$$

Since $\mathcal{O}^U_{\Phi_p}$ is compact, the function f assumes a minimum at a point, say $Ad_{u_0}\Phi_p = \Phi'_p$. That means that for every $\lambda \in \mathfrak{u}$ we have

$$\frac{d}{dt}|_{t=0} \mathcal{K}(Ad_{exp(t\lambda)}\Phi'_p \ , \ y) = 0 \ . \qquad (3.42)$$

This implies

$$\mathcal{K}([\lambda, \Phi'_p] \ , \ y) = \mathcal{K}(\lambda \ , [\Phi'_p, y]) = 0$$

for every $\lambda \in \mathfrak{u}$. Since $\Phi'_p \ , \ y \in \mathfrak{p}$, we have $[\Phi'_p, y] \in \mathfrak{u}$, and so from 3.42, $[\Phi'_p, y] \in \mathfrak{u} = 0$, since the restriction of $\mathcal{K}$ on $\mathfrak{u}$ is non-degenerate. Now, because of the maximality of $\mathfrak{h}_{\mathfrak{p}}$ and the choice of y, we finally get $\Phi'_p \in \mathfrak{h}_{\mathfrak{p}}$. Recall that the group $W_{\mathfrak{p}}$ is generated by the reflections s_α associated to the roots $\alpha \in \Delta_{\mathfrak{p}}$ i.e., those which do not identically vanish on the subspace $\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h}$. Suppose $\alpha \in \Delta^+ \setminus \Delta_{\mathfrak{p}}$. Then

$$s_\alpha(x) = x - 2\langle \alpha \ , \ x \rangle \cdot \alpha^* = x$$

for every $x \in \mathfrak{h}_{\mathfrak{p}}$, since $\langle \alpha \ , \ x \rangle = 0$ in this case. From this it is clear that the W-orbit of Φ'_p is actually the $W_{\mathfrak{p}}$-orbit, which proves the lemma. □

Recall now that we constructed the integrals H_j of the system $(T^*G^{\mathbb{C}}, \omega_{can}, H)$ by

$$q_i(\Phi(z)) = \sum_{j=0}^{d_i} H_{(j+\sum_{k=1}^{i-1}(2d_k+1))} \cdot z^j \ , \tag{3.43}$$

where $\{q_1, \ldots q_r\}$ is a basis of the ring $I^{G^{\mathbb{C}}}$ of the polynomial invariants on $\mathfrak{g}^{\mathbb{C}}$. By $\{q_1^W, \ldots, q_r^W\}$ we are going to denote the basis of the W-invariant polynomials I^W on $\mathfrak{h}$ corresponding to the basis $\{q_1, \ldots, q_r\}$ via the Chevalley's isomorphism, and by $\{q_1^{W_p}, \ldots, q_p^{W_p}\}$ a basis of the ring $I^{W_{\mathfrak{p}}}$ of $W_{\mathfrak{p}}$-invariants on $\mathfrak{h}_{\mathfrak{p}}$.

We are going to divide the constraints $F_j(H_1, \ldots, H_n)$ into two subsets, the description of the first being obvious and that of the second one somewhat more complicated.

The algebra of polynomials on a vector space V can be identified by the symmetric part $S^*(V^*)$ of the graded tensor algebra $T^*(V^*)$ on the space V^* dual to V. Suppose that $V = V_1 \oplus V_2$. Then, for each degree d, we have the identity

$$S^d(V^*) = \bigoplus_{a=0}^{d} S^{d-a}(V_1^*) \otimes S^a(V_2^*) \ .$$

Applying this to the space $\mathfrak{h} = \mathfrak{h}_{\mathfrak{u}} \oplus \mathfrak{h}_{\mathfrak{p}}$, we can write the polynomial $q_i^W(X) = q_i^W(X_u + X_p)$ in the form

$$q_i^W(X) = \sum_{a=0}^{d_i} g^{d_i - a}(X_p) \cdot h^a(X_u) = g^{d_i}(X_p) + \sum_{a=1}^{d_i} g^{d_i - a}(X_p) \cdot h^a(X_u) \ .$$

If the $\mathfrak{u}$-component of X is equal to zero, that is if $X = X_p$, the above equation assumes a simpler form

$$q_i^W(X_p) = g^{d_i}(X_p) \ .$$

Clearly, g^{d_i} is an element of $I^{W_{\mathfrak{p}}}$ in this case, and it can therefore be expressed as a polynomial function of the elements $\{q_1^{W_p}, \ldots, q_p^{W_p}\}$ of our chosen basis of $I^{W_{\mathfrak{p}}}$. But we can actually be more definite about these polynomial functions, which can assume a very simple form, provided we make a suitable choice of the basis $I^{W_{\mathfrak{p}}}$.

In [Hu 1] the following two facts about the Poincare polynomials $W(t)$ of the reflection groups can be found:

(i) Let W be a group generated by a set of reflections of the vector space $\mathfrak{h}$. Then its Poincare polynomial can be written in the form

$$W(t) = \prod_{i=1}^{r} \frac{t^{d_i} - 1}{t - 1} ,$$

where $d_i's$ are the degrees of the elements in a basis of W-invariant polynomials on $\mathfrak{h}$.

(ii) Let $W_{\mathfrak{p}}$ be a subgroup of W generated by some subset of the reflections which generate W. Then

$$W(t) = W_{\mathfrak{p}}(t) \cdot W_{\mathfrak{r}}(t) ,$$

where $W_{\mathfrak{p}}(t)$ is the Poincare polynomial of $W_{\mathfrak{p}}$ and $W_{\mathfrak{r}}(t)$ a suitable rational function in t.

For the proof of the above facts see [Hu 1], pages 84 and 123. From *(i)* and *(ii)* it follows that

$$W_{\mathfrak{p}} = \prod_{i=1}^{p} \frac{t^{d'_i} - 1}{t - 1}$$

where $\{d'_1, \ldots, d'_p\} \subset \{d_1, \ldots, d_r\}$. That means that the degrees of the $W_{\mathfrak{p}}$-invariants are a subset of the degrees of W-invariants.

Consider now the mapping $Q : \mathfrak{h} \to \mathfrak{h}/W$ defined by

$$Q(X_1, \ldots, X_r) = (q_1^W(X_1, \ldots, X_r) \ , \ldots , \ q_r^W(X_1, \ldots, X_r)) .$$

This is a surjective mapping from $\mathfrak{h}$ to the space of W-orbits $\mathfrak{h}/W$. As we have seen, the W-orbit of an element $X \in \mathfrak{h}_{\mathfrak{p}}$ is actually a $W_{\mathfrak{p}}$-orbit and therefore the restriction $Q/\mathfrak{h}_{\mathfrak{p}}$ is a surjective map from $\mathfrak{h}_{\mathfrak{p}}$ on the space of $W_{\mathfrak{p}}$-orbits $\mathfrak{h}_{\mathfrak{p}}/W_{\mathfrak{p}}$. Let the coordinates $(X_1, \ldots, X_r)$ of $\mathfrak{h}$ be arranged so that $(X_1, \ldots, X_p)$ span the subspace $\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h}$, and consider the Jacobian matrix

$$\mathcal{J} = \begin{pmatrix} \frac{\partial q_1^W}{\partial X_1} & \frac{\partial q_1^W}{\partial X_2} & \cdots & \frac{\partial q_1^W}{\partial X_r} \\ \frac{\partial q_2^W}{\partial X_1} & \frac{\partial q_2^W}{\partial X_2} & \cdots & \frac{\partial q_2^W}{\partial X_r} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q_r^W}{\partial X_1} & \frac{\partial q_r^W}{\partial X_2} & \cdots & \frac{\partial q_r^W}{\partial X_r} \end{pmatrix} \tag{3.44}$$

of the mapping Q. After permuting the rows in the way, where for $i = 1, \dots p$ the polynomial q_i^W will have the degree d_i', and when evaluated at the points with $X_{p=1} = \dots = X_r = 0$, the principal $(p \times p)$-minor of $\mathcal{J}$ will become the Jacobian matrix of the restriction $Q_{/W_{\mathfrak{p}}}$. Because of the surjectivity of $Q_{/W_{\mathfrak{p}}}$ this minor is non-degenerate, so by the Jacobian criterion for algebraic independence proved in [Hu 1], page 63, the polynomials $q_1^W/_{\mathfrak{h}_{\mathfrak{p}}}, \dots, q_r^W/_{\mathfrak{h}_{\mathfrak{p}}}$ are algebraically independent. Since they have the right degrees, they can be taken as a basis $\{q_1^{W_p}, \dots, q_p^{W_p}\}$ of the ring $I^{W_{\mathfrak{p}}}$ of $W_{\mathfrak{p}}$-invariants on $\mathfrak{h}_{\mathfrak{p}}$. By lemma 10 we have $W \cdot X_p = W_{\mathfrak{p}} \cdot X_p$ for every $X_p \in \mathfrak{h}_{\mathfrak{p}}$, therefore it is clear that

$$q_i^W(X_p) = 0 \tag{3.45}$$

for every $i = p, \dots, r$ and for every $X_p \in \mathfrak{h}_{\mathfrak{p}}$.

Choose now a point in the intersection $\mathcal{O}_{\Phi(z)} \cap \mathfrak{h}$ for each $z \in \mathbb{CP}^1$ and denote it by $X_\Phi(z)$. By the Chevalley isomorphism we have $q_i(\Phi(z)) = q_i^W(X_\Phi(z))$ for every $i = 1, \dots r$. Recall the expression 3.41 giving $\Phi(z_0) \in \mathfrak{p}^{\mathbb{C}}$ for $z_0 = \pm i$. From lemma 10 we then get

$$q_i(\Phi(z_0)) = q_i^W(X_\Phi(z_0)) = g^{di}(X_\Phi(z_0))$$

for every $i = 1, \dots r$, since by lemma 10 the element $X_\Phi(z_0)$ lies in the subspace $\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h}$. Putting the above expression in 3.45 we get

$$q_i^W(X_\Phi(z_0)) = 0 \quad , \quad i = p, \dots, r \, .$$

This, together with 3.43, finally gives

$$\sum_{j=0}^{d_i} (z_0)^j H_{(j + \sum_{k=1}^{i-1}(2d_k+1))} = 0 \, , \quad i = p, \dots, r \, , \tag{3.46}$$

providing us with the first subset of constraints announced above.

We are going to describe the second subset of constraints using some facts about the Jacobian determinant $J = det\mathcal{J}$ of the map $Q : \mathfrak{h} \to \mathfrak{h}/W \cong \mathbb{C}^r$. Let $\Phi(z) \in H^0(\mathbb{CP}^1;\, \mathfrak{g}^{\mathbb{C}} \otimes \mathcal{O}(2))$ be a solution of the Lax equation 3.26. Let $W * H^0(\mathbb{CP}^1; \mathfrak{h} \otimes \mathcal{O}(2))$ be the space of objects $\mathcal{X}$, such that $\mathcal{X}_z$ is the W-orbit of $X(z)$ for some element $X \in H^0(\mathbb{CP}^1; \mathfrak{h} \otimes \mathcal{O}(2))$ Let $\mathcal{X}_\Phi \in W * H^0(\mathbb{CP}^1; \mathfrak{h} \otimes \mathcal{O}(2))$ be such that $\mathcal{X}_\Phi = \mathcal{O}_{\Phi(z)} \cap \mathfrak{h}$ for every $z \in \mathbb{CP}^1$. Since $J(X^i_{\Phi(z)}) = J(X^j_{\Phi(z)})$ for every pair of branches $X^i_{\Phi(z)}, X^j_{\Phi(z)}$, we get a well defined element $J(\mathcal{X}_\Phi) \in H^0(\mathbb{CP}^1; \mathcal{O}(m))$, where $m = dim(\mathfrak{g}^{\mathbb{C}}) - rank(\mathfrak{g}^{\mathbb{C}})$.

Later in the text we shall prove that the section $J(\mathcal{X}_\Phi) \in H^0(\mathbb{CP}^1; \mathcal{O}(m))$ is an invariant of motion given by Lax equation. We will show this by proving that $J(\mathcal{X}_\Phi)$ is the discriminant of the spectral curve associated with the Lax equation.

Let $z_o = \pm i$. For our purposes it will be convenient to write the polynomial $J(\mathcal{X}_\Phi(z))$ in the form

$$J(\mathcal{X}_\Phi(z)) = \prod_{i=1}^{m}(\chi_i - (z - z_0))\ . \tag{3.47}$$

It is clear from the above lemma that the functions χ_i for $i = 1, \ldots, m$ are independent invariants of our motion. We are going to compare the above expression with the well-known factorization of the Jacobian of W-invariants

$$J(X) = \prod_{\alpha\in\Delta^+} \lambda_\alpha(X)\ , \tag{3.48}$$

where λ_α denotes a polynomial of degree 1 given by the formula

$$\lambda_\alpha(X) = \langle X\ ,\ \alpha\rangle\ .$$

Evidently we have $\lambda_{-\alpha} = -\lambda_\alpha$. The zero sets of these polynomials are the hyperplanes $H_\alpha = \{X \in \mathfrak{h};\ \langle X\ ,\ \alpha\rangle = 0\}$. Note that H_α are the mirrors of the respective reflections s_α. For the proof of 3.48 see e.g. [Hu 1] or [He 2]. Let $\Theta : \mathfrak{h} \to \mathfrak{h}$ be the involution having $\mathfrak{h}_\mathfrak{u}$ as the $(+1)$-eigenspace, and $\mathfrak{h}_\mathfrak{p}$ as the (-1)-eigenspace, i.e. the restriction of the Cartan involutionan involution on $\mathfrak{h}$. Denote by α^Θ the element in $\mathfrak{h}^*$ defined by $\alpha^\Theta(X) = \alpha(\Theta X)$, and let $H_{\alpha+\alpha^\Theta} \subset \mathfrak{h}$ be the hyper-space, which is the zero set of the 1-form $(\alpha + \alpha^\Theta) \in \mathfrak{h}^*$. Observe that if $\alpha \in \Delta_\mathfrak{p}$ vanishes identically on $\mathfrak{h}_\mathfrak{u}$, then $\alpha^\Theta = -\alpha$ and therefore $(\alpha + \alpha^\Theta) \equiv 0$.

Lemma 11 *Let $\mathfrak{g}^\mathbb{C} = \mathfrak{u}^\mathbb{C} \oplus \mathfrak{p}^\mathbb{C}$ be a Cartan decomposition, and let*

$$\mathfrak{h} = \mathfrak{h}_\mathfrak{u} \oplus \mathfrak{h}_\mathfrak{p} \subset \mathfrak{g}^\mathbb{C}$$

be a Cartan sub-algebra, such that $\mathfrak{h}_\mathfrak{p} = \mathfrak{h} \cap \mathfrak{p}^\mathbb{C}$ is a maximal Abelian subspace in $\mathfrak{p}^\mathbb{C}$. Then

$$\mathfrak{h}_\mathfrak{p} = (\bigcap_{\alpha\in\Delta^+\setminus\Delta_\mathfrak{p}} H_\alpha) \cap (\bigcap_{\alpha\in\Delta'_\mathfrak{p}} H_{\alpha+\alpha^\Theta})\ ,$$

where $\Delta'_\mathfrak{p}$ denotes the subset of roots $\alpha \in \Delta_\mathfrak{p}$, such that $\alpha|_{\mathfrak{h}_\mathfrak{u}} \not\equiv 0$.

Proof: Let first $X \in \mathfrak{h}_\mathfrak{p}$ be an arbitrary element. Then for every $\alpha \in \Delta^+ \setminus \Delta_\mathfrak{p}$ we have $\langle\alpha, X\rangle = 0$ and so by the definition $X \in H_\alpha$ Take any $\alpha \in \Delta'_\mathfrak{p}$. The definition of α^Θ gives

$$\langle(\alpha + \alpha^\Theta), X\rangle = \langle\alpha, X\rangle + \langle\alpha, \Theta X\rangle = \langle\alpha, (X + \Theta X)\rangle = 0\ ,$$

since obviously $(X + \Theta X) = 0$ for every $X \in \mathfrak{h}_\mathfrak{p}$. Hence we have $X \in H_{\alpha+\alpha^\Theta}$ and therefore

$$\mathfrak{h}_\mathfrak{p} \subset (\bigcap_{\alpha\in\Delta^+\setminus\Delta_\mathfrak{p}} H_\alpha) \cap (\bigcap_{\alpha\in\Delta'_\mathfrak{p}} H_{\alpha+\alpha^\Theta})\ .$$

Let now $X \in \mathfrak{h}$ be an element with a non-zero $\mathfrak{h}_{\mathfrak{u}}$-component X_u. Since the elements $\alpha \in \Delta^+$ span the space $\mathfrak{h}^*$, there exists a root $\alpha \in \Delta^+$, such that $\langle \alpha, X_u \rangle \neq 0$. This root can be either an element of $\Delta^+ \setminus \Delta_{\mathfrak{p}}$ or of $\Delta'_{\mathfrak{p}}$. In the first case

$$\langle \alpha, X \rangle = \langle \alpha, X_u \rangle \neq 0 \, ,$$

while in the second

$$\langle (\alpha + \alpha^{\Theta}), X \rangle = 2\langle \alpha, X_u \rangle \neq 0 \, .$$

This implies $X \notin (\bigcap_{\alpha \in \Delta^+ \setminus \Delta_{\mathfrak{p}}} H_\alpha) \cap (\bigcap_{\alpha \in \Delta'_{\mathfrak{p}}} H_{\alpha+\alpha^\Theta})$ and completes the proof of the lemma. □

Let $\Delta''_{\mathfrak{p}}$ denote the set $\Delta_{\mathfrak{p}} \setminus \Delta'_{\mathfrak{p}}$. It is proved in [He 1], page 222, that $-\alpha^\Theta \in \Delta_{\mathfrak{p}}$ whenever $\alpha \in \Delta_{\mathfrak{p}}$. Note that $\alpha^\Theta = -\alpha$ if $\alpha \in \Delta''_{\mathfrak{p}}$. All the roots from the set $\Delta'_{\mathfrak{p}}$ can be arranged into pairs α, α^Θ, with $\alpha \neq \alpha^\Theta$. This allows us to decompose the factorization 3.48 in the following way:

$$J(X) = (\prod_{\alpha \in \Delta^+ \setminus \Delta_{\mathfrak{p}}} \lambda_\alpha(X)) \cdot (\prod_{\alpha \in \Delta'_{\mathfrak{p}},\ \alpha \neq \alpha^\Theta} (\lambda_\alpha \lambda_{-\alpha^\Theta})(X)) \cdot (\prod_{\alpha \in \Delta''_{\mathfrak{p}}} \lambda_\alpha(X)) \, . \qquad (3.49)$$

Let now $\Phi(z)$ be a solution of the Lax equation 3.26 and let $\mathcal{X}_\Phi(z) \in \mathcal{O}_\Phi \cap \mathfrak{h}$ be the section of $\mathfrak{h} \otimes \mathcal{O}(2)$ as above. Then each branch of $\mathcal{X}_\Phi(z_0)$ lies in $\mathfrak{h}_{\mathfrak{p}}$ for $z_0 = \pm i$. From lemma 11 and using the expressions 3.47 and 3.49 we can deduce the following two facts about the polynomial $J(\mathcal{X}_\Phi(z))$

<u>*(a)*</u> Let $\mathcal{A}$ denote the number of roots lying in the set $\Delta^+ \setminus \Delta \mathfrak{p}$. Then the point z_0 is the zero of degree $\mathcal{A}$ of the polynomial $J(\mathcal{X}_\Phi(z))$. Renumbering the factorization $J(\mathcal{X}_\Phi(z)) = \prod_{i=1}^N (\chi_i - (z - z_0))$ if necessary, this gives us $\mathcal{A}$ constraints

$$\chi_i = 0 \quad , \quad i = 1, \dots, \mathcal{A} \, . \qquad (3.50)$$

<u>*(b)*</u> It is clear from the definition (and also from lemma 11) that for every $\alpha \in \Delta'_{\mathfrak{p}}$ we have $\alpha|_{\mathfrak{h}_{\mathfrak{p}}} = -\alpha^\Theta|_{\mathfrak{h}_{\mathfrak{p}}}$, and therefore

$$\lambda_\alpha(X_p) = -\lambda_{\alpha^\Theta}(X_p) \, . \qquad (3.51)$$

Let $X_\Phi(z)$ be one branch of $\mathcal{X}_\Phi(z)$. Then $X_\Phi(z) = X_1 + (z - z_0)X_2 + (z - z_0)^2 X_3$, where $X_1 \in \mathfrak{h}_{\mathfrak{p}}$, and we can write

$$\lambda_\alpha(X_\Phi(z)) = \langle \lambda_\alpha, X_1 \rangle + (z - z_0)\langle \lambda_\alpha, X_2 \rangle + (z - z_0)^2 \langle \lambda_\alpha, X_3 \rangle \, .$$

Taking a suitable indexation in the expression 3.47, we can write

$$\lambda_\alpha(X_\Phi(z)) = (\chi_{(N-i)} - (z - z_0)) \cdot (\chi_{(N-(i+1))} - (z - z_0)) \, ,$$

and thus $\chi_{(N-i)} \cdot \chi_{(N-(i+1))} = \langle \lambda_\alpha, X_1 \rangle$. Let $\mathcal{B}$ denote the number of roots lying in $\Delta'_{\mathfrak{p}}$. In the same way (again taking care of the indices) we get $\chi_{(N-\mathcal{B})+i)} \cdot \chi_{((N-\mathcal{B})+i+1)} = \langle \lambda_{\alpha^\Theta}, X_1 \rangle$. From 3.51 we then finally get

$$\chi_{(N-i)} \cdot \chi_{(N-(i+1))} = -\chi_{((N-\mathcal{B})+i)} \cdot \chi_{((N-\mathcal{B})+i+1))} \ , \quad i = 1, 3, \dots, \frac{1}{2}\mathcal{B} \tag{3.52}$$

It is clear from the expression 3.49, that $\mathcal{B}$ is an even number.

Recall now lemma 8, where it was claimed

$$\mathfrak{u}^{\mathbb{C}} = \mathfrak{h}_{\mathfrak{u}} \oplus \bigoplus_{\Delta^+ \backslash \Delta_{\mathfrak{p}}} (\mathfrak{g}^{\alpha} + \mathfrak{g}^{-\alpha}) \oplus \sum_{\alpha \in \Delta_{\mathfrak{p}}} \mathbb{C}(x_\alpha + \Theta(x_\alpha)) \quad .$$

This, together with the expressions 3.46, 3.50, and 3.52 applied to the cases $z_0 = i$ and $z_0 = -i$, establishes the proof of the following proposition.

Proposition 28 *Let $T_i : I \to \mathfrak{g}^{\mathbb{C}}, i = 1, 2, 3$ satisfy the Nahm's equations*

$$\dot{T}_i = \sum \varepsilon_{i,j,k} [T_j, T_k]$$

and the additional conditions

$$\begin{array}{rcl} T_1, T_3 : & I \longrightarrow & \mathfrak{p}^{\mathbb{C}} \\ T_2 : & I \longrightarrow & \mathfrak{u}^{\mathbb{C}} \end{array} \ .$$

Then there exist $dim_{\mathbb{C}}\ \mathfrak{u}^{\mathbb{C}}$ algebraically independent relations (constraints) among the integrals H_i defined by 3.32 . Explicitly, these constraints are given by the following set of equations.

(i) $\sum_{j=0}^{d_i} (\pm i)^j H_{(j + \sum_{k=1}^{i-1} d_k)} = 0 \ , \quad i = p, \dots, r \ ,$

(ii) $\chi_i^\epsilon = 0 \quad , \quad i = 1, \dots, \mathcal{A} \ , \quad \epsilon = 1, 2 \ .$

(iii) $\chi^\epsilon_{(N-i)} \cdot \chi^\epsilon_{(N-(i+1))} = -\chi^\epsilon_{((N-\mathcal{B})+i)} \cdot \chi^\epsilon_{((N-\mathcal{B})+i+1))} \ , \quad i = 1, 3, \dots, \frac{1}{2}\mathcal{B} \ , \quad \epsilon = 1, 2$

Here χ_i^ϵ are the invariants defined by the expression $J(\mathcal{X}_\Phi(z)) = \prod_{i=1}^m (\chi_i^\epsilon - (z - z_\epsilon))$, where $z_\epsilon = i$ and $z_\epsilon = -i$.

□

Remark 8 *Clearly, the invariants χ_i^1 and χ_i^2 are not independent, but the relations (ii) and (iii) by contrast are.*

3.4.2

In this subsection we are going to show what happens to the integrals of the "master system" on $G^{\mathbb{C}}$ when we reduce it to the system on the homogeneous space $\widetilde{\mathcal{H}} = G^{\mathbb{C}}/\widetilde{G}$, where $\widetilde{G}$ is some real form of the complex group $G^{\mathbb{C}}$. This, together with the results obtained in the previous subsection will provide the full set of restrictions that one has to impose on the integrals of the system on $G^{\mathbb{C}}$ to get those of the system on the symmetric space $M = G/U$.

The system on $\widetilde{\mathcal{H}} = G^{\mathbb{C}}/\widetilde{G}$ corresponds to the Nahm's equations for the functions T_i taking values in the real form $\widetilde{\mathfrak{g}} \subset \mathfrak{g}^{\mathbb{C}}$. In this case the functions 3.24 will have the form

$$\begin{array}{ccccc} \alpha & = & (T_2 + iT_3) & : I \longrightarrow \mathfrak{g}^{\mathbb{C}} \\ \gamma = \widetilde{\tau}(\alpha) & = & (T_2 - iT_3) & : I \longrightarrow \mathfrak{g}^{\mathbb{C}} \\ \beta & = & -2iT_1 & : I \longrightarrow \mathfrak{g}^{\mathbb{C}} \end{array} \tag{3.53}$$

Where $\widetilde{\tau}$ is the real structure of $G^{\mathbb{C}}$ belonging to $\widetilde{G}$ .The solution of the first equation of 3.27 will then be

$$\mathcal{A} = Ad_{(g,\widetilde{\tau}(g))}(\alpha, \widetilde{\tau}(\alpha))$$

and therefore the solution of the corresponding variational problem will be of the form

$$h(t) = (g, \widetilde{\tau}(g)) \cdot \tau((g, \widetilde{\tau}(g))^{-1} = (\widetilde{h}, \widetilde{h}^{-1}) : I \longrightarrow (G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}{}_r.$$

Here $\widetilde{h} = g\widetilde{\tau}(g)^{-1} \in \widetilde{\mathcal{H}} \subset G^{\mathbb{C}} \cong (G^{\mathbb{C}} \times G^{\mathbb{C}})/G^{\mathbb{C}}{}_r$. Let again $p_1 = 0, p_4 = \infty$. In this case, the element Φ appearing in the Lax equation 3.26 becomes

$$\Phi(z) = \alpha + z\beta + z^2\widetilde{\tau}(\alpha)\ .$$

In addition we have $\widetilde{\tau}(\beta) = -\beta$. Consider Φ as an element in $H^0(\mathbb{CP}^1; ad(P)\otimes\mathcal{O}(2)) = H^0(\mathbb{CP}^1; \mathfrak{g}^{\mathbb{C}} \otimes K^{-1})$. Let

$$\sigma : T\mathbb{CP}^1 \longrightarrow T\mathbb{CP}^1$$

be the lifting of the antipodal map of $\mathbb{CP}^1$ on its tangent bundle given by the formula

$$(z, \frac{\partial}{\partial z}) \longrightarrow (-\frac{1}{\bar{z}}, -(\bar{z}^2\frac{\partial}{\partial \bar{z}}))$$

Define the involution Σ on $H^0(\mathbb{CP}^1; ad(P) \otimes \mathcal{O}(2))$ by $\Sigma = [(\widetilde{\tau} \otimes id) \circ \sigma]$. Then we immediately see:

$$\Sigma(\Phi) = \Phi.$$

This allows us to construct an involution ρ on the space of Hamiltonians

$$\bigoplus_{i=1}^{r} H^0(\mathbb{CP}^1; K(D)^{d_i}) = \bigoplus_{i=1}^{r} H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))$$

which corresponds to Σ, in the sense that its fixed points will be the Hamiltonians of the system, where Φ_t is a fixed point of Σ for every $t \in I$. We will define ρ on the summands $H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))$ of the space of Hamiltonians by

$$\rho\Big(q_j(\Phi(z))\Big) = q_j\Big(\Sigma(\Phi(z))\Big).$$

The integrals of the system on $\widetilde{\mathcal{H}}$ will be given by the fixed point set of ρ and will therefore become real functions.

3.4.3

The constraints obtained in the previous two sections might seem somewhat obscure, but nevertheless they do have a suggestive geometric description in terms of the spectral curve. The discussion of the spectral curve demands a case by case approach depending on the type of the Lie group involved, but the peculiarities stemming from the particular Lie group involved do not show up in the description of the constraints.

We also compute the dimension of the linear system $|S|$ of the spectral curve. Here, as in the previous chapter we confine ourselves to the case $G^{\mathbb{C}} = SL(n;\mathbb{C})$. The treatment of the situations with other classical groups would again be a more or less straightforward application of the results from [Hi 1].

First, we recall the definition of the spectral curve. We are going to start with the spectral curve for the system $(T^*(G^{\mathbb{C}} \times G^{\mathbb{C}})/G_r^{\mathbb{C}}, \omega_{can}, H)$. As in previous chapter, we construct the ruled surface $\mathcal{R} = \mathbb{P}(\mathcal{O}(2) \oplus \mathbb{C})$ and denote by $\widetilde{\mathcal{O}}(2) \to \mathcal{R}$ the pull-back of the bundle $p : \mathcal{O}(2) \to \mathbb{CP}^1$ by the natural projection p. Let $\Phi \in H^0(\mathbb{CP}^1; adP \otimes \mathcal{O}(2))$ be an arbitrary section. For every symmetric polynomial q_i of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ we get a holomorphic section $q_i(\Phi) \in H^0(\mathbb{CP}^1; \mathcal{O}(2d_i))$, where as before $d_i = deg(q_i)$. Use the projection p again to get the pull-backs $q_i^*(\Phi) \in H^0(\mathcal{R}; \widetilde{\mathcal{O}}(2d_i))$.

Let $w \in H^0(\mathcal{R}; \widetilde{\mathcal{O}}(2))$ be the tautological section and

$$\mathcal{Q} = w^{d_r} + \sum_{i=1}^{r} w^{(d_r - d_i)} \cdot q_i^*(\Phi)$$

an element in $H^0(\mathcal{R}; \widetilde{\mathcal{O}}(2d_r))$. The spectral curve $S(\Phi) \subset \mathcal{R}$ of the element $\Phi \in H^0(\mathbb{CP}^1; adP \otimes \mathcal{O}(2))$ is the zero divisor of the section $\mathcal{Q}$.

By a similar argument as in [Hi 1] it can be seen that the linear system of $\mathcal{Q}$ is without base points, therefore by Bertini's theorem $S(\Phi)$ is a smooth curve. It is a d_r-sheeted ramified covering of $\mathbb{CP}^1$.

Recall that $\Phi_t(z) : \mathbb{R} \to T^*\mathcal{M}_{D^d}$ is a solution of our variational problem if and only if it is a solution of the Lax equation 3.29. As we have seen, $q_i(\Phi_t(z)) \equiv q_i(\Phi_0(z))$ for every solution $\Phi_t(z)$ of 3.29, so the section $\mathcal{Q}$ and the spectral curve S are invariants of the system $(T^*\mathcal{M}_{D^d}, \omega_{can}, H)$, as was already observed in the previous chapter. Denote by $|S|$ the linear system of the curve S in the surface $\mathcal{R}$, and define the mapping

$$\Omega : T^*\mathcal{M}_{D^d} \longrightarrow |S|$$

by $\Omega(\Phi) = S(\Phi)$. The components of this map are the integrals of our variational system, and their relation to the previously described integrals is easily seen by using the Vietta rules.

We are going to calculate the dimension $dim|S|$, using a method slightly different from the one applied in subsection 2.4.1. The following simple lemma is central to this subsection. Denote by $J(\Phi)$ the Jacobian determinant $J(\mathcal{X}_\Phi)$ and let $m = dim\mathfrak{g}^{\mathbb{C}} - rank\mathfrak{g}^{\mathbb{C}}$.

Lemma 12 *The section $J(\Phi)^2 \in H^0(\mathbb{CP}^1; \mathcal{O}(2m))$ is the discriminant of the ramified covering $S(\Phi) \to \mathbb{CP}^1$. This means that $z \in \mathbb{CP}^1$ is a zero of $J(\Phi)^2$ if and only if it is a ramification point of $S(\Phi) \to \mathbb{CP}^1$. Clearly all the zeros of $J(\Phi)^2$ are at least double.*

Proof: In subsection 3.4.1 we have shown how to assign an element $\mathcal{X}_\Phi \in W * H^0(\mathbb{CP}^1; \mathfrak{h} \otimes \mathcal{O}(2))$ to the element $\Phi \in T^*\mathcal{M}_{D^d}$. A point $z \in \mathbb{CP}^1$ is a zero of $J(\Phi)$ if and only if one (and hence at least two) branches of $\mathcal{X}_\Phi$ lies in $H_\alpha(z)$, where $H_\alpha(z)$ is a wall of a Weyl chamber in the fibre $(\mathfrak{h} \otimes \mathcal{O}(2))_z \cong \mathfrak{h}$. This can be seen immediately from the factorization $J(X) = \prod_{\alpha \in \Delta^+} \lambda_\alpha(X)$.

Let $a \in \mathfrak{g}^{\mathbb{C}}$ be an arbitrary element. Then the set of the zeros of the polynomial

$$Q(w) = w^{d_r} + \sum_{i=1}^{r} w^{(d_r - d_i)} \cdot q_i(a)$$

taken in some suitable order can be thought of as the coordinates of a point in $\mathbb{C}^{d_r}$. This space contains the Cartan sub-algebra $\mathfrak{h} \cong \mathbb{C}^r$ as a subspace, and the point a actually lies in $\mathfrak{h} \subset \mathbb{C}^{d_r}$. The particular inclusion of $\mathfrak{h}$ into $\mathbb{C}^{d_r}$ depends on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ in question. In the case of $\mathfrak{s}l(n;\mathbb{C})$ we have $\mathfrak{h} = \{(a_1, \ldots, a_n) \in \mathbb{C}^n \ ; \ \sum_{i=1}^n a_i = 0\}$, in the case of $\mathfrak{s}o(2n)$, $\mathfrak{h} = \{(a_1, \ldots, a_{2n}) \in \mathbb{C}^{2n} \ ; \ a_i = -a_{i+1}\}$, etc... In any case the point a lies in at least one of the walls $H_\alpha \subset \mathfrak{h}$ if and only if at least two of the coordinates $(a_1, \ldots, a_{d_r})$ are equal. This is so because in such a case the point a always has a nontrivial stabiliser W_a with respect to the action of the Weyl group W on $\mathfrak{h}$. Using the Chevalley homomorphism, we can apply these remarks to the zero divisor $S(\Phi)$ of the section $\mathcal{Q} \in H^0(\mathcal{R}; \widetilde{\mathcal{O}}(2d_r))$.

Let $z_0 \in \mathbb{CP}^1$ be a zero of $J(\Phi)^2$. What remains to be proved is that the order of zero z_0 is equal to the ramification number of the covering $S(\Phi) \to \mathbb{CP}^1$ at z_0. The proof can be found in [Gr], but we repeat it here for the sake of completeness.

Let $w_1(z), \ldots, w_k(z)$ be k of the zeroes of $\mathcal{Q} \in H^0(\mathcal{R}; \mathcal{O}(2d_r))$ coalescing in the point z_0 and suppose they are labelled in such a way that they permute cyclically when z encircles z_0. Then we can choose a local coordinate ζ around z_0, such that $w_j = e^{j2\pi i/k} \cdot \zeta^{1/j}$. We have

$$\prod_{j<l} (e^{j2\pi i/k}\zeta^{1/k} - e^{l2\pi i/k}\zeta^{1/k}) = const \cdot \zeta^{k-1} \, .$$

Taking into account all the ramifications above the point z_0, we see that the order of vanishing of $J(\Phi)^2(z)$ at z_0 indeed coincides with the ramification number of $S(\Phi)$ at this point.

The above lemma allows us to prove the following proposition.

Proposition 29 *Let $\Phi \in T^*\mathcal{M}_{D^d}$ be an arbitrary element, $S(\Phi)$ its spectral curve and $\widetilde{\mathcal{O}}(2d_r) = L \to \mathcal{R}$ the line bundle with the holomorphic section $\mathcal{Q}(\oplus)$ having $S(\Phi)$ as the zero locus. Then*

$$h^0(\mathcal{R}; L) = 1 + 3 \cdot d_r + dim(\mathfrak{g}^{\mathbb{C}}) - rank(\mathfrak{g}^{\mathbb{C}}) \, . \tag{3.54}$$

Proof: Denote by $L \to \mathcal{R}$ the line bundle $\widetilde{\mathcal{O}}(2d_r)$. Then $S(\Phi)$ is the zero locus of the section $\mathcal{Q} \in H^0(\mathcal{R}; L)$. Recall the version of the Riemann-Roch formula for the linear system of a curve used in the previous chapter.

$$\chi(L) = \chi(\mathcal{O}_{\mathcal{R}}) + \frac{(L \cdot L - L \cdot K_{\mathcal{R}})}{2} \, .$$

The formula 2.40 from subsection 2.4.1 gives $\chi(\mathcal{O}_{\mathcal{R}}) = 1$ in our case. Subtracting this from the genus given by the adjunction formula

$$g(S(\Phi)) = 1 + \frac{(L \cdot L + L \cdot K_{\mathcal{R}})}{2}$$

we get

$$\chi(L) = g(S(\Phi)) - L \cdot K_{\mathcal{R}} \, .$$

On the other hand, we can obtain the genus from the Riemann-Hurwitz formula

$$g(S(\Phi)) = 1 - d_r + \frac{R}{2} \, ,$$

where R is the ramification index of the covering $S(\Phi) \to \mathbb{CP}^1$. The formula 2.42 gives $L \cdot K_{\mathcal{R}} = -4 \cdot d_r$, so we have

$$\chi(L) = 1 + 3 \cdot d_r + \frac{R}{2} \, .$$

From lemma 12 we get

$$\chi(L) = 1 + 3 \cdot d_r + dim(\mathfrak{g}^{\mathbb{C}}) - rank(\mathfrak{g}^{\mathbb{C}}) .$$

An application of the Kodaira vanishing theorem gives $h^0(\mathcal{R}; L) = \chi(L)$, which completes the proof. □

In the case, where $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}l(n;\mathbb{C})$ 3.54, becomes

$$h^0(\mathcal{R}; L) = n^2 + 2n + 1 .$$

Suppose now that S is the spectral curve of a solution $\Phi_t(z) = \alpha_t + z\beta_t + z^2\gamma_t$ of the Lax equation 3.29. Then $S(\Phi_t) \equiv S(\Phi_0)$. The points of S intersecting the fibres $\mathcal{O}(2)_0$ and $\mathcal{O}(2)_\infty$ are the spectra of the elements $\alpha_0, \gamma_0 \in \mathfrak{g}^{\mathbb{C}}$. Since α_0 and γ_0 are the data of our variational problem, S lies in the linear subsystem in $\mathcal{R}$ consisting of the curves having the values at 0 and ∞ fixed as described above. Taking $\mathfrak{s}l(n;\mathbb{C})$ as the structure group, restricts us further to the subsystem of codimension 1 whose elements are the curves satisfying the condition

$$\sum_{w_i \in S(\Phi) \cap \mathcal{O}(2)_z} w_i = 0 \ , \quad for\ every \quad z \in \mathbb{CP}^1 .$$

From this we finally get

Proposition 30 *Let S be the spectral curve of the Lax equation*

$$\dot{\Phi}_t(z) = [\frac{d}{dz}\Phi_t(z) \ , \ \Phi_t(z)] ,$$

where $\Phi_t(z) : I \to T^*\mathcal{M}_{D^d}$, and $\mathcal{M}_{D^d}$ is the moduli space of the parabolic $SL(n;\mathbb{C})$-bundles over $\mathbb{CP}^1$ with two double marked points. Then we have

$$dim|S| = dim(\mathfrak{s}l(n;\mathbb{C})) = n^2 - 1 ,$$

where $|S|$ denotes the linear subsystem consisting of the spectral curves of the above Lax equation, and the dimensions are complex.

□

The effect of the constraints from subsection 3.4.2 on the linear system $|S|$ is quite straightforward. Recall the involution

$$\sigma : T\mathbb{CP}^1 \longrightarrow T\mathbb{CP}^1$$

defined as the lifting of the antipodal map of the sphere to its tangent bundle. Using σ and a real structure τ on $\mathfrak{g}^{\mathbb{C}}$, we defined the involution ρ by

$$\rho(q_i(\Phi(z))) = q_i(\Sigma(\Phi)) ,$$

where $\Sigma = [(\tau \otimes id) \circ \sigma] : H^0(\mathbb{CP}^1; adP \otimes \mathcal{O}(2)) \longrightarrow H^0(\mathbb{CP}^1; adP \otimes \mathcal{O}(2))$. Let now $\mathcal{Q} = w^{d_r} + \sum_{i=1}^r q_i^*(\Phi(z)) \in H^0(\mathcal{R}; \mathcal{O}(2d_r))$ with the zero divisor $S_\mathcal{Q}$ and let $\mathcal{Q}_\Xi$ be $\mathcal{Q}_\Xi = w^{d_r} + \sum_{i=1}^r w^{(d_r-d_i)}\rho(q_i(\Phi))^*$ with the zero divisor denoted by $S_{\mathcal{Q}_\Xi}$ This gives the involution

$$\Xi_\tau : |S| \longrightarrow |S| \, , \tag{3.55}$$

defined by $\Xi(S_\mathcal{Q}) = S_{\mathcal{Q}_\Xi}$. The involution Ξ_τ is a real structure on the the complex space $|S|$ depending on the real structure τ on $\mathfrak{g}^\mathbb{C}$. The following claim is now obvious

Let $\Phi_t : I \to T^*\mathcal{M}_{D^d}$ be a solution of our variational system on the real homogeneous space $\widetilde{\mathcal{H}} = G^\mathbb{C}/\widetilde{G}$. Then the corresponding spectral curve $S(\Phi)$ lies in the fixed-point set $|S|_{\widetilde{\tau}}$ of the real structure $\Xi_{\widetilde{\tau}} : |S| \longrightarrow |S|$, $\widetilde{\tau}$ being the real form of $\mathfrak{g}^\mathbb{C}$ corresponding to the real form $\widetilde{\mathfrak{g}} \subset \mathfrak{g}^\mathbb{C}$. The (real) coordinates of the point $S(\Phi) \in |S|_{\widetilde{\tau}}$ are the integrals of the motion.

Finally, we will interpret the constraints from subsection 3.4.1 in terms of the linear system of the spectral curve, i.e. we will determine the linear subsystem $|S|_M \subset |S|_{\widetilde{\tau}} \subset |S|$ whose elements are the spectral curves corresponding to our variational problem on the symmetric space $M \subset \widetilde{\mathcal{H}}$. The constraints from 3.4.1 involve the "shape" of the spectral curve at the points $i, -i \in \mathbb{CP}^1$ Since these two points are antipodal we see from the above observations that the situation at one of the points determines the situation at the other, allowing us to keep only one of them, say i, in mind.

First, recall the point *(i)* of proposition 28. In a more compact form these constraints are

$$q_j(\Phi_t(i)) = 0$$

for $j \in \mathcal{I}$, where $\mathcal{I}$ is some set containing $r - p$ indices, p being the rank of the symmetric space M. The spectral curves satisfying these conditions are the zero divisors of the sections

$$\mathcal{Q}_\mathcal{I} = w^{d_r} + \sum_{j \notin \mathcal{I}} w^{d_r - d_j} \cdot (q_j)^*(\Phi) \, .$$

Denote the resulting subsystem of $|S|_\tau$ by $|S|_\mathcal{I}$. The curves lying in $|S|_\mathcal{I}$ can be characterised as follows. Let s_j denote the basic symmetric polynomial in d_r variables of degree $d_j = deg(q_j)$. Let S be a spectral curve and let $w(i)_j, j = 1, \dots, d_r$ be the points in the intersection $S \cap T_i\mathbb{CP}^1$ labelled in some arbitrary order. Then $S \in |S|_\mathcal{I}$ if and only if

$$s_j(w(i)_1, \dots, w(i)_{dr}) = 0 \, , \quad j \in \mathcal{I} \, . \tag{3.56}$$

The point *(ii)* of 28 is responsible for fixing a part of the ramification of the covering $S \to \mathbb{CP}^1$. In lemma 12 we have seen that the section $J(\Phi)^2$ is the discriminant of the covering $S(\Phi) \to \mathbb{CP}^1$. Let S be the spectral curve of a variational system satisfying the condition *(ii)*, 3.4.1. Then S is an element of the linear subsystem $|S|_{\mathcal{R}} \subset |S|_\tau$ which consists of the ramified coverings $S \to \mathbb{CP}^1$ having a fixed ramification of degree $\mathcal{A}$ over the point $i \in \mathbb{CP}^1$. Recall that $\mathcal{A}$ is the number of the roots $\alpha \in \Delta^+$ which vanish identically on the subspace $\mathfrak{h}_{\mathfrak{p}}$ of our chosen Cartan sub-algebra $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$.

The last set of constraints *(iii)* in 3.4.1 has the form

$$\chi_{(N-i)} \cdot \chi_{(N-(i+1))} = -\chi_{((N-\mathcal{B}))} \cdot \chi_{((N-\mathcal{B}+i+1))} , \tag{3.57}$$

where $J(\Phi(z)) = \prod_{i=1}^{m}(\chi_i - (z-i))$. Reindexing the zeroes if necessary, this gives the comparison of pairs of factors with degree 2 in $J(\Phi(z))$. It is clear from the point *(b)* of the proof of proposition 3.4.1, that these factors are precisely those of the form $\lambda_\alpha(X_\Phi)$, which in turn coincide with the ones appearing in the local expression $J(\Phi(z)) = \prod_{1\le i\le j\le d_r}(w_i(z) - w(_j(z))$ of $J(\Phi(z))$ as the discriminant of the covering $S(\Phi) \to \mathbb{CP}^1$ which was discussed in lemma 12. In short

$$(\chi_i - (z-i)) \cdot (\chi_{(i+1)} - (z-i)) = \lambda_{\alpha_r}(\mathcal{X}_\Phi) = (w_k(z) - w_l(z))$$

for a suitable choice if indices. Evaluating the above at $z = i$, we see from the condition 3.57, that there are $\frac{1}{2}\mathcal{B}$ quadruples of points $(w_j(i), \ldots, w_{(j+3)}(i))$ from the intersection of $S(\Phi)$ with the fibre $\mathcal{O}(2)_i$, such that $(w_j(i) - w_{(j+1)}(i)) = (w_{(j+2)}(i) - w_{(j+3)}(i))$.

We summarize the above in the following observation. Recall that $\mathcal{A}$ is the number of the roots $\alpha \in \Delta^+ \setminus \Delta_{\mathfrak{p}^{\mathbb{C}}}$ of the roots on $\mathfrak{h} \in \mathfrak{g}^{\mathbb{C}}$ vanishing identically on $\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h}$, $\mathcal{B}$ is the number of the roots in the subset $\Delta_{\mathfrak{p}'} \subset \Delta_{\mathfrak{p}}$, such that $\alpha \neq \alpha^\Theta$. The set $\mathcal{I}$ is the subset of those indices i from $\{1, \ldots, r\}$ for which the invariant functions q_i^W vanish on $\mathfrak{h}_{\mathfrak{p}} \subset \mathfrak{h}$.

Proposition 31 *Let the Hamiltonian system (T^*M, ω_{can}, H) describe the motion of a particle on the symmetric space M governed by the Hamiltonian*

$$H = \|p\|^2 + \mathcal{K}(Ad_q(\beta), \widetilde{\beta}) ,$$

and let $S \subset \mathcal{R}$ be the spectral curve of this system. The coordinates of the S in the linear system $|S|$ of the divisors equivalent to S in $\mathcal{R}$ form a redundant set of Poisson commuting first integrals of our Hamiltonian system. The system $|S|$ contains a linear subsystem $|S|_M$ with $dim|S|_M = dimM$. This subsystem is determined by the following data:

(a) Every element $S \in |S|_M$ is a fixed point of the involution $\Xi_{\widetilde{\tau}} : |S| \to |S|$, defined in 3.55

(b) Let $(w_1(i), \ldots, w_{d_r}(i))$ *be the points in the intersection* $S_i = S \cap \mathcal{O}(2)_i$*, and let* s_i *be the elementary symmetric functions with* $deg(s_j) \in \mathcal{I}$*. Then*

$$s_j(w_1(i), \ldots, w_{d_r}(i)) = 0 \quad , for\ j \in \mathcal{I}.$$

(c) The ramified covering $S \to \mathbb{CP}^1$ *has a fixed (partial) ramification of degree* $\mathcal{A}$ *at the point* $i \in \mathbb{CP}^1$ *.*

(d) There are $\frac{1}{2}\mathcal{B}$ *quadruples of points in* S_i *satisfying the conditions of the form*

$$(w_j(i) - w_{(j+1)}(i)) = (w_{(j+2)}(i) - w_{(j+3)}(i)) \, .$$

□

3.5 Examples

The family of the Hamiltonian systems (T^*M, ω_{can}, H), where M is an arbitrary Riemannian symmetric space includes a vast variety of concrete examples of integrable Hamiltonian systems. We are going to mention here only a few of them. The first obvious application of our main result is the following proposition

Proposition 32 *The geodesic motion on an arbitrary Riemannian symmetric space is a completely integrable Hamiltonian system in the Liouville sense.*

Proof: Let $M = G/U$. We have shown that for every $\beta \in \mathfrak{g}$, the Hamiltonian system (T^*M, ω_{can}, H), with

$$H = \|p\|_M^2 + \mathcal{K}(Ad_q(\beta).\widetilde{\beta})$$

is an integrable system. Putting $\beta = 0$ in the above expression sets the potential part to zero and hence gives the Hamiltonian of the free particle on M, i.e. the geodesic motion. □

This fact was already proved by Mischenko in [Mi] using a different approach.

Another example which follows immediately from theorem 7 is the following

Proposition 33 *Let* G *be a real Lie group. Then the system* (T^*G, ω_{can}, H) *describes the motion of a particle* G *under the influence of the potential* $V(g) = \mathcal{K}(Ad_g(\beta), \beta)$*, with* $\beta \in \mathfrak{g}$*. This system is completely integrable.*

□

Let T_β be the stabiliser of β with respect to the coadjoint action of G on $\mathfrak{g}^*$. Then T_β acts on (T^*G, ω_{can}, H), so we can form the symplectic quotient. Denoting by $\mu : T^*G \to \mathfrak{t}^*_\beta$ the corresponding moment map, we get the following corollary.

Corollary 2 *The Hamiltonian system $(T^*\mathcal{O}_\beta, \omega_{can}, \widetilde{H})$ is completely integrable.*

□

3.5.1

C. Neumann's system

In this paragraph we will study the system that will justify the title of the present chapter. Our symmetric space in this case is going to be the n-dimensional standard sphere S^n. We are going to establish the following fact:

Proposition 34 *In the case where the Riemannian symmetric space M is the n-dimensional sphere $\mathcal{S} \cong S^n$, our Hamiltonian system (T^*M, ω_{can}, H) coincides with the classical C. Neumann's system describing the harmonic motion constrained to the sphere.*

Proof: The C. Neumann system is given by $(T^*\mathbb{R}^n, \omega_{can}, H_N)$, where

$$H_N(q,p) = \|p\|^2 - \langle Aq, q\rangle \ ,$$

with the constraints

$$\|q\|^2 = 1 \quad , \quad \langle q, p\rangle = 0 \ .$$

Here A is a symmetric $n \times n$-matrix . (Without the loss of generality we can assume that A is a diagonal matrix.) This system was first described in [Ne], but it subsequently became quite a popular topic touched by many authors. (See e.g. [A-vM 1], [Mo 1], [Mu] [Uh], and many others.)

First we describe the sphere S^n in terms of the involutions following the recipe given in proposition 19. For every n we have $S^n = SO(n+1)/SO(n)$. Obviously $SO(n+1)^{\mathbb{C}} = SO(n+1;\mathbb{C})$. The two relevant real structures of $SO(n+1;\mathbb{C})$ are then

$$\tau(h) = \bar{h}, \quad \widetilde{\tau}(h) = J\bar{h}J \ ,$$

where J is the matrix of the form

$$J = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} .$$

The involution $\widetilde{\sigma}$ from proposition 19 is then given by $\widetilde{\sigma}(h) = J\bar{h}^{-1}J = J\bar{h}^{T}J$. Applying proposition 19 to this situation we then get the following description: The sphere S^n is isomorphic to the subspace $\mathcal{S}$ of the group $SO(n+1;\mathbb{C})$ consisting of all the elements h of the form

$$h = \begin{pmatrix} r_{0,0} & -r_{0,1} & -r_{0,2} & \dots & -r_{0,n} \\ r_{0,1} & r_{1,1} & r_{1,2} & \dots & r_{1,n} \\ r_{0,2} & r_{1,2} & r_{2,2} & \dots & r_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{0,n} & r_{1,n} & r_{2,n} & \dots & r_{n,n} \end{pmatrix} .$$

Here all the numbers $r_{i,j}$ are real and the columns (or the rows) of the matrix h form an orthonormal frame in the space $\mathbb{R}^{n+1}$.

In order to prove that the system $(T^*\mathcal{S}, \omega_{can}, H)$ is the C. Neumann system, it is enough to show that the potential energy part $V(h) = \mathcal{K}(Ad_h\beta_0, \widetilde{\sigma}(\beta))$ of the Hamiltonian H coincides with the function $\langle Ax, x\rangle$ defined on the sphere S^n. The kinetic parts of H_N and H are obviously the same.

Let β be a real matrix. Observe that $h^{-1} = h^T = JhJ$. Since on $SO(n+1;\mathbb{C})$ the Killing form is given by $\mathcal{K}(x,y) = -Tr(x\cdot y)$, we get

$$V(h) = \mathcal{K}(Ad_h\beta, -J\beta J) = Tr(h\cdot\beta')^2 \,, \tag{3.58}$$

where $\beta' = \beta\cdot J$. Real matrices of dimension $(n+1)\times(n+1)$ can be thought of as vectors in the space $\mathbb{R}^{(n+1)^2}$. The usual Euclidean scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{(n+1)^2}$ is given by

$$\langle\widetilde{X}, \widetilde{Y}\rangle = Tr(X\cdot Y^T)\,, \tag{3.59}$$

where $\widetilde{X} = (u_0, \dots, u_n) \in \mathbb{R}^{(n+1)^2}$ and u_i is the i-th row of the matrix X. From this it is clear that $SO(n+1) \subset S^{(n+1)^2-1} \subset \mathbb{R}^{(n+1)^2}$, since $Tr(X\cdot X^T) = Tr(X\cdot X^{-1}) \equiv (n+1)$ for every $X \in SO(n+1)$. (The radius of our sphere is $\sqrt{n+1}$.) The space $\mathcal{S}$ described above is a subspace of $SO(n+1)$, therefore we have $\mathcal{S} \cong S^n \subset S^{(n+1)^2-1} \subset \mathbb{R}^{(n+1)^2}$, i.e. $\mathcal{S}$ lies in $S^{(n+1)^2-1}$ as an equatorial sphere. Comparing 3.58 and 3.59, we get

$$V(h) = Tr(h\cdot\beta')^2 = Tr(\beta' h\beta'\cdot h) = \langle\widetilde{\beta' h\beta'}, \widetilde{h^T}\rangle\,. \tag{3.60}$$

The mapping $\widetilde{h} \mapsto \widetilde{\beta h\beta}$ is of course linear and is self-adjoint with respect to the Euclidean scalar product if an only if $\beta^T = \beta$. Indeed:

$$\langle\widetilde{\beta h\beta}, \widetilde{k}\rangle = Tr(\beta h\beta\cdot k^T) = Tr(h\cdot\beta k^T\beta) = \langle\widetilde{h}, \widetilde{\beta^T k\beta^T}\rangle\,. \tag{3.61}$$

Now recall that $h^T = JhJ$. Putting this into 3.60 will give

$$V(h) = \langle \widetilde{\beta' h \beta'}, \widetilde{JhJ} \rangle = \langle \widetilde{(J\beta J)h\beta} \ , \ \widetilde{h} \rangle \ ,$$

since $J^T = J$. Let $\mathcal{B}$ denote the matrix of the linear transformation $\widetilde{h} \mapsto \widetilde{(J\beta J)h\beta}$ From 3.61 we can then immediately conclude that the linear mapping

$$\mathcal{B} : \mathbb{R}^{(n+1)^2} \longrightarrow \mathbb{R}^{(n+1)^2}$$

is self-adjoint, i.e. $\mathcal{B}^T = \mathcal{B}$ if β satisfies the conditions

$$J\beta^T J = \beta \ . \tag{3.62}$$

The space $\mathcal{S}$ lies in the subspace of $\mathcal{H}' \subset \mathbb{R}^{(n+1)^2}$ consisting of the matrices α of the form

$$\alpha = \begin{pmatrix} a_{00} & a \\ -a^T & \chi \end{pmatrix} \ ,$$

where $a = (a_{01}, \dots a_{0n})$ and χ is a symmetric $n \times n$ matrix. One can directly check that $\mathcal{B}$ preserves the subspace $\mathcal{H}'$. Let $\mathcal{A} \cong \mathbb{R}^{(n+1)}$ denote the smallest linear subspace in $\mathcal{H}'$ that contains the sphere $\mathcal{S} \cong S^n$. In other words $\mathcal{A} = \{\lambda \cdot h \ ; \ h \in \mathcal{S} \ , \ \lambda \in \mathbb{R}\}$. Let

$$\pi : \mathcal{H}' \longrightarrow \mathcal{A}$$

be the projection orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle$. Since $\mathcal{B}$ is symmetric, so is the composite map

$$\mathcal{B}' = \pi \circ \mathcal{B} : \mathcal{A} \longrightarrow \mathcal{A} \ .$$

For every $h \in \mathcal{A}$ we have

$$\langle \mathcal{B}'\widetilde{h}, \widetilde{h} \rangle = \langle \mathcal{B}\widetilde{h}, \pi^T \widetilde{h} \rangle = \langle \mathcal{B}\widetilde{h}, \widetilde{h} \rangle \ , \tag{3.63}$$

which enables us to express the function 3.60 as a quadratic form

$$V(h) = Tr(h \cdot \beta') = \langle \mathcal{B}'\widetilde{h}, \widetilde{h} \rangle \tag{3.64}$$

on the space $\mathcal{A} \cong \mathbb{R}^{(n+1)}$, provided that the matrix β satisfies the conditions 3.62. □

The following remark is in place here. For any pair of integers (p, q) such that $p + q = n$, the real Grassmanian $Gr_{(p,q)}(\mathbb{R})$ can be obtained as a homogeneous space $\mathcal{G}_{(p,q)} = SO(n)/SO(p) \times SO(q)$. Following the procedure described in subsection 3.1.1, we can represent $\mathcal{G}_{(p,q)}$ as a fixed point set of the involutions $\widetilde{\sigma} : SO(n, \mathbb{C}) \to SO(n, \mathbb{C})$ and $\tau : SO(n, \mathbb{C}) \to SO(n, \mathbb{C})$. Here $\widetilde{\sigma}(a) = \widetilde{\tau}(a^{-1})$, and $\widetilde{\tau}$, τ are the real structures of $SO(n, \mathbb{C})$ corresponding to the real forms $SO(p, q)$ and $SO(n)$ respectively. So we get $\mathcal{G}_{(p,q)} \subset SO(n) \subset S^{(n+1)^2}$. From the discussion above it is clear that the Hamiltonian

H of our general system (T^*M, ω_{can}, H) for the case $M = \mathcal{G}_{(p,q)}$ is the restriction to $\mathcal{G}_{(p,q)} \subset S^{(n+1)^2}$ of the Hamiltonian

$$H(q,p) = \|p\|^2 - \langle \mathcal{B}q, q \rangle$$

defined on $T^*S^{(n+1)^2}$, giving the C. Neumann system on $S^{(n+1)^2}$. Therefore we can think of the system $(T^*\mathcal{G}_{p,q}, \omega_{can}, H)$ as of a special case of a C. Neumann system, subject to the additional constraints keeping the motion of the particle on the subspace $\mathcal{G}_{p,q} \subset S^{(n+1)^2}$. Some of these constraints are precisely those described in subsection 3.4.1.

The spherical pendulum

The spherical pendulum is a classical mechanical system describing the motion of a particle confined to the sphere $S^2 \in \mathbb{R}^3$ under the influence of the gravitational force. Hence the phase space of this system is T^*S^2, where $S^2 = \{(q_1, q_2, q_3) \in \mathbb{R}^3;\ q_1^2 + q_2^2 + q_3^2 = 1\}$ and the Hamiltonian is given by

$$H(q,p) = \|p\|^2 + q_3 \,.$$

Periodic motions of this system were discovered already by Huygens. A detailed treatment of the spherical pendulum was carried out by Duistermaat in [Du].

Recall proposition 33, describing the motions on the real semi-simple groups. Let the group G be the group of rotations $SO(3)$. There is an isometry

$$\mathcal{I} : (\mathfrak{so}(3), \mathcal{K}) \to (\mathbb{R}^3, \langle \cdot, \cdot \rangle) \,,$$

where $\mathcal{K}$ is the Killing form on the Lie algebra $\mathfrak{so}(3)$ and $\langle \cdot, \cdot \rangle$ the standard Euclidean structure on $\mathbb{R}^3$. Let $\beta \in \mathfrak{so}(3)$ correspond to the point $e_3 = (0,0,1)$ under this isometry. We are going to prove the following proposition.

Proposition 35 *Let $U(1)_\beta \subset SO(3)$ be the stabiliser of $\beta \in \mathfrak{so}(3)^*$. Then the action of $U(1)_\beta$ on $T^*SO(3)$ preserves the Hamiltonian H. Let $\mu : T^*SO(3) \to i\mathbb{R}$ be the corresponding moment map. The reduced system $(\mu^{-1}(0)/U(1)_\beta, \widetilde{\omega}, \widetilde{H})$ is the spherical pendulum.*

Proof: Since $SO(3)/U(1)_\beta = S^2$ and since $\mu^{-1}(0)/U(1)_\beta = T^*(SO(3)/U(1)_\beta)$, we see that the phase space of the reduced space is indeed T^*S^2. What remains to be shown is that the potential part $\widetilde{V}$ of the reduced Hamiltonian $\widetilde{H}$ is equal to $V_{sp}(q) = q_3$.

Under the isometry $\mathcal{I}$ the adjoint action of $SO(3)$ on $\mathfrak{so}(3)$ translates into the usual action of $SO(3)$ as the rotations on $\mathbb{R}^3$. From this we see that the 2-sphere $\{q = A(e_3) \in \mathbb{R}^3; A \in SO(3)\}$ is precisely the quotient $SO(3)/U(1)_\beta$. Moreover

$$\mathcal{K}(Ad_A(\beta), \beta) = \langle A(e_3), e_3 \rangle = \langle q, e_3 \rangle = q_3 \,,$$

which proves the proposition. □

3.5.2

Motion on a sphere in a quartic potential and motions on projective spaces

First we are going to give a short description of the embeddings of the sphere S^n and the projective space $\mathbb{RP}^n$ into the group $SO(n+1)$.

We have already seen that $S^n \cong \mathcal{S} \subset SO(n+1)$. From the description of $\mathcal{S}$ in the previous subsection we see:

$$\mathcal{S} = exp(\mathfrak{p}) ,$$

where $\mathfrak{p}$ is the subspace of $\mathfrak{so}_{(n+1)}$ consisting of the elements of the form

$$\alpha = \begin{pmatrix} 0 & a \\ a^T & 0 \end{pmatrix}$$

with $a = (x_1, \dots x_n)$. The general form of $h = exp(\alpha)$ is

$$h = \begin{pmatrix} \cos\|a\| & \frac{x_1}{\|a\|}\sin\|a\| & \dots & \frac{x_n}{\|a\|}\sin\|a\| \\ -\frac{x_1}{\|a\|}\sin\|a\| & 1 - \frac{x_1^2}{\|a\|^2} + \frac{x_1^2}{\|a\|^2}\cos\|a\| & \dots & -\frac{x_1 x_2}{\|a\|^2} + \frac{x_1 x_2}{\|a\|^2}\cos\|a\| \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{x_n}{\|a\|}\sin\|a\| & -\frac{x_1 x_2}{\|a\|^2} + \frac{x_1 x_2}{\|a\|^2}\cos\|a\| & \dots & 1 - \frac{x_n^2}{\|a\|^2} + \frac{x_n}{\|a\|^2}\cos\|a\| \end{pmatrix} .$$

It can then be seen (e.g. putting $a = (x_1, 0 \dots, 0)$ in the above expression) that $h = exp(\alpha)$ is an element of $SO(n+1)$ characterised by the following data. Let $e_1 = (1, 0, \dots, 0)$

(i) $h(e_1) = (cos\|a\|, \frac{x_1}{\|a\|}\sin\|a\|, \dots, \frac{x_n}{\|a\|}\sin\|a\|)$, i.e. h rotates the vector e_1 by the angle $\|a\|$ within the plane P spanned by e_1 and the vector $(0, x_1, \dots, x_n)$.

(ii) h fixes all the vectors orthogonal to the plane P

To every element $h \in \mathcal{S}$ we can assign a reflection of the space $\mathbb{R}^{(n+1)}$ in a unique way. Denote by X the vector $h(e_1) = (\cos\|a\|, \frac{x_1}{\|a\|}\sin\|a\|, \dots, \frac{x_n}{\|a\|}\sin\|a\|) \in S^n$ and let $k \in O(n+1)$ be the reflection of $\mathbb{R}^{(n+1)}$ through the hyper-plane orthogonal to the vector X. The reflection k is given by the formula

$$\widetilde{k}(y) = y - 2\langle X, y\rangle \cdot X ,$$

and therefore by the matrix

$$\widetilde{k} = I - 2X^T X .$$

It is easily seen that we have the following relation of matrices

$$h^2 = \widetilde{k} \cdot J .$$

The map $h \mapsto k = \widetilde{k} \cdot J$ is a double cover mapping the sphere $\mathcal{S} \cong S^n$ onto the projective space $\mathbb{RP}^n$. We will denote this concrete realization of $\mathbb{RP}^n$ as a subspace of $SO(n+1)$ by $\mathcal{RP}^n$. As a homogeneous space, $\mathcal{RP}^n$ is isomorphic to the quotient $SO(n+1)/S(O(1)\times O(n))$. In terms of the involutions τ and $\widetilde{\sigma}$ there is no difference between the cases $SO(n+1)/SO(n)$ and $SO(n+1)/S(O(1)\times O(n))$ on the level of Lie algebras. But the difference does occur on the level of the groups. In the first case we take as a fixed point set of the Cartan involution

$$h \mapsto J \cdot h \cdot J$$

only the component containing the identity, i.e. $SO(n) \subset SO(n+1)$, while in the second case we take both components of $S(O(1)\times O(n)) \cong U(n) \subset SO(n+1)$.

Now we are going to describe our system (T^*M, ω_{can}, H) for the case where the symmetric space M is the real projective space $\mathbb{RP}^n \cong \mathcal{RP}^n$. Our approach will be the following. Above we have constructed the mapping

$$\widetilde{\kappa} : S^n \cong \mathcal{S} \longrightarrow \mathbb{RP}^n \cong \mathcal{RP}^n$$

given by $\widetilde{\kappa}(h) = h^2 \cdot J$. The element $h \in \mathcal{S}$ is uniquely determined by the vector $X = (\cos\|a\|, \frac{x_1}{\|a\|}\sin\|a\|, \ldots, \frac{x_n}{\|a\|}\sin\|a\|)$, which enables us to construct a new mapping

$$\kappa : S^n \longrightarrow \mathbb{RP}^n \cong \mathcal{RP}^n ,$$

defined by the formula

$$\kappa(X) = k = (I - 2X^T X) \cdot J . \tag{3.65}$$

Clearly, this is the usual antipodal map. Using this relation, we are going to describe a mechanical system on the sphere S^n which will then descend on the system on the projective space $\mathbb{RP}^n$ since it is invariant with respect to the appropriate $\mathbb{Z}_2$-action. More precisely, we are going to prove the following proposition.

Proposition 36 *Let the Hamiltonian system $(T^*S^n, \omega_{can}, H_{(4)})$ on the sphere S^n be given by the Hamiltonian*

$$H_{(4)} = \|p\|^2 + 4\langle\beta' q, \beta' q\rangle - 4\langle\beta' q, q\rangle^2 . \tag{3.66}$$

Then the mapping $d^*\kappa : T^*\mathbb{RP}^n \cong T^*\mathcal{RP}^n \to T^*S^n$, where κ is the antipodal map, lifts the solutions of the system $(T^*S^{2n+1}, \omega_{can}, H_{(4)})$ to the solutions of the system $(T^*\mathcal{RP}^n, \omega_{can}, H)$.

In particular the system $(T^*S^n, \omega_{can}, H_{(4)})$, which describes the motion of a particle on the sphere S^n under the influence of the quartic potential $H_{(4)}$, is a completely integrable Hamiltonian system.

Proof: It will be more convenient to work initially with the Lagrangians rather than with the Hamiltonians. The Lagrangian of our system on $\mathcal{RP}^n$ is given by

$$L(k) = \|k^{-1}\dot{k}\|^2_{\mathcal{RP}} + \mathcal{K}(Ad_k(\beta), J\bar{\beta}J) . \tag{3.67}$$

Comparing the metric $\langle \cdot, \cdot \rangle_{\mathcal{RP}}$ to $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ and then observing that $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ is the standard metric on S^n, we see that the pull back of $\langle \cdot, \cdot \rangle_{\mathcal{RP}}$ is the standard metric on S^n. Since κ is the antipodal map, the metric $\langle \cdot, \cdot \rangle_{\mathcal{RP}}$ is the canonical one on $\mathbb{RP}^n$.

Using 3.65 , 3.67 and the fact $\kappa^*\langle \cdot, \cdot \rangle_{\mathcal{RP}} = \langle \cdot, \cdot \rangle_{S^n}$, we are going to compute the Lagrangian $L_{S^n}(X)$ of the above mentioned system on S^n. We start with the kinetic term

$$\|k^{-1} \cdot \dot{k}\|^2 = \mathcal{K}(k^{-1}\dot{k}, k^{-1}\dot{k}) = -Tr(JkJ\dot{k} \cdot JkJ\dot{k}) .$$

Denoting $k = (I - 2X^TX)J = (I - 2A)J$, we then get

$$\|k^{-1} \cdot \dot{k}\|^2 = -4 \cdot Tr((I - 2A)\dot{A} \cdot (I - 2A)\dot{A}) = -4 \cdot Tr(\dot{A}^2 - 2A\dot{A}^2 + 4A\dot{A}A\dot{A}) .$$

Since $a = (I - 2X^TX) = (I - 2A)$ is a reflection, we have $a^2 = Id$ and therefore $A^2 = A$. Differentiation gives $Tr(\dot{A}) = Tr(2A\dot{A})$, and thus

$$\|k^{-1} \cdot \dot{k}\|^2 = -16 \cdot Tr(A\dot{A}A\dot{A}) .$$

Differentiating $A^2 = A$ we get $A\dot{A} = \dot{A} - \dot{A}A$ and from this $16 \cdot Tr(A\dot{A}A\dot{A}) = 16 \cdot Tr(A\dot{A} - A\dot{A}^2)$, and using $\frac{1}{2}Tr(\dot{A}) = Tr(A\dot{A})$ again, we finally get

$$\|k^{-1} \cdot \dot{k}\|^2 = 8 \cdot Tr(\dot{A}^2 - \dot{A}) .$$

Recalling that $A = X^TX$ and using $(\dot{X}^TX\dot{X}^TX)^T = X^T\dot{X}X^T\dot{X}$ we write

$$Tr(\dot{A}^2) = 2 \cdot Tr((\dot{X}^TX)^2 + (X^T\dot{X})(\dot{X}^TX)) .$$

A straightforward calculation gives

$$Tr((X^T\dot{X})(\dot{X}^TX)) = \|X\|^2 \cdot \|\dot{X}\|^2 ,$$

and

$$Tr(\dot{X}^TX)^2 = \langle X, \dot{X}\rangle^2 .$$

In addition we also have

$$Tr(\dot{A}) = 2Tr(\dot{X}^TX) = 2\langle X, \dot{X}\rangle .$$

Summarizing the above, we get the following expression for the kinetic term:

$$\|k^{-1} \cdot \dot{k}\|^2 = 16 \cdot (\|X\|^2 \cdot \|\dot{X}\|^2 + \langle X, \dot{X}\rangle^2 - \langle X, \dot{X}\rangle) \tag{3.68}$$

Next, we express the potential $\mathcal{K}(Ad_k(\beta), J\bar{\beta}J)$ of the Lagrangian $L(k)$ in terms of X. Using the fact $k^{-1} = JkJ$ and taking β real we get

$$\mathcal{K}(Ad_k(\beta), J\bar{\beta}J) = Tr(k(\beta J) \cdot k(\beta J)) \, .$$

From the expression $k = (I - X^T X) \cdot J$ it then follows

$$\mathcal{K}(Ad_k(\beta), J\bar{\beta}J) = Tr((I - X^T X)\beta' \cdot (I - X^T X)\beta') \, ,$$

where $\beta' = J\beta J$. Expanding this gives

$$\mathcal{K}(Ad_k(\beta), J\bar{\beta}J) = Tr(\beta'^2) - 4 \cdot Tr(X^T X \cdot \beta'^2) + 4 \cdot Tr(X^T X \beta' X^T X \beta') \, .$$

For the second term in the above expression we have $Tr(X^T X \beta'^2) = Tr(\beta'^2 \cdot X^T X) = \langle \beta'^2 X, X \rangle$. Assume without the loss of generality that β' is diagonal. Then we have $X^T \beta' X = \langle \beta' X, X \rangle$, and therefore $Tr(X^T X \beta' X^T X \beta') = \langle \beta' X, X \rangle^2$. Putting the terms together, we get the following expression for the potential term

$$\mathcal{K}(Ad_k(\beta), J\bar{\beta}J) = Tr(\beta'^2) - 4\langle \beta' X, \beta' X \rangle + 4\langle \beta' X, X \rangle^2 \, . \tag{3.69}$$

Inserting the kinetic part 3.68 and the potential part 3.69 into the Lagrangian 3.67 we finally get

$$L_{s^n}(X) = 16 \cdot (\|X\|^2 \cdot \|\dot{X}\|^2 + \langle X, \dot{X} \rangle^2 - \langle X, \dot{X} \rangle) + Tr(\beta'^2) - 4\langle \beta' X, \beta' X \rangle + 4\langle \beta' X, X \rangle^2 \, .$$

Since the particle moves on the unit sphere S^n, it is subject to the constraints

$$\|X\|^2 = 1 \quad , \quad \langle X, \dot{X} \rangle = 0$$

which yields the following expression for the Lagrangian L_{S^n}:

$$L_{S^n} = \|\dot{X}\|^2 + Tr(\beta'^2) - 4\langle \beta' X, \beta' X \rangle + 4\langle \beta' X, X \rangle^2 \, .$$

Applying the Legendre transformation and neglecting the constant part $Tr(\beta'^2)$, which does not affect the corresponding force field, but only determines the zero level of the energy, we get the Hamiltonian (3.66)

$$H_{(4)}(q, p) = \|p\|^2 + 4\langle \beta' q, \beta' q \rangle - 4\langle \beta' q, q \rangle^2 \, ,$$

describing the motion of a particle in a quartic potential. □

Consider now the complex projective space $\mathbb{CP}^n$. As a homogeneous space it is represented as $\mathbb{CP}^n = SU(n+1)/S(U(1) \times U(n))$. The realization of $\mathbb{CP}^n$ as a fixed point set of a pair of involutions of $SL(n+1\,;\mathbb{C})$ described in subsection 3.1.1 will be denoted by $\mathcal{CP}^n \subset SU(n+1)$. An approach analogous to the one used in proposition 36 will be applied in the description of the integrable Hamiltonian system $(T^*\mathcal{CP}^n, \omega_{can}, H)$.

Lemma 13 *Every element $k \in \mathcal{CP}^n$ is a linear map of the form*

$$k = (I - 2Z^*Z) \cdot J \,,$$

for some $Z \in S^{2n+1} \subset \mathbb{C}^{(n+1)}$. The mapping

$$\vartheta : S^{2n+1} \longrightarrow \mathcal{CP}^n \cong \mathbb{CP}^n$$

given by $\vartheta(Z) = (I - 2Z^*Z) \cdot J$ is the usual Hopf fibration.

Proof: First we show that $\vartheta(S^{2n+1}) \subset \mathcal{CP}^n$. The elements of $\alpha \in \mathcal{CP}^n$ are characterised by the property $J \cdot \alpha \cdot J = \alpha^{-1}$, i.e. $(\alpha \cdot J)^2 = I$. Since $k \cdot J = (I - 2Z^*Z)$ is the complexified reflection through the hyper-plane orthogonal to Z, we see that $(k \cdot J)^2 = I$, which proves $\vartheta(S^{2n+1}) \subset \mathcal{CP}^n$. Since S^{2n+1} is compact, the image $\vartheta(S^{2n+1})$ is closed in $\mathcal{CP}^n$. On the other hand it is also open. This follows from the fact that ϑ is a submersion. Its derivative at the point $Z = (1, 0, \dots 0)$ is given by $d\vartheta(\dot{Z}) = 4 \cdot \dot{Z}$, so it is surjective. By the homogeneity this is true at any other point Z. Finally, we have

$$\vartheta(e^{i\varphi} \cdot Z) = (I - 2Z^* e^{-i\varphi} e^{i\varphi} Z) \cdot J = (I - 2Z^*Z) \cdot J \,.$$

Summarizing, the map $\vartheta : S^{2n+1} \longrightarrow \mathcal{CP}^n \cong \mathbb{CP}^n$ is a $U(1)$ invariant surjective submersion, hence it is indeed the Hopf fibration. □

Let now $\pi : S^{2n+1} \longrightarrow \mathbb{CP}^n$ be the Hopf fibration given in the usual form, that is by $\pi(z_0, z_1, \dots, z_n) = [z_0, z_1, \dots, z_n]$, where $[z_0, z_1, \dots z_n]$ are the homogeneous coordinates. The metric on the sphere S^{2n+1} is inherited from the hermitian product on $\mathbb{C}^{(n+1)}$. At an arbitrary $z \in S^{2n+1}$ this product gives the orthogonal decomposition $\mathbb{R} \cdot z \oplus T_z S^{2n+1}$. In addition, the tangent space of the $U(1)$-action at z is the subspace $\mathbb{R} \cdot iz$ giving a further orthogonal decomposition

$$\mathbb{C}^{(n+1)} = \mathbb{R} \cdot z \oplus \mathbb{R} \cdot iz \oplus H_z \,.$$

The distribution of the subspaces $H_z \subset T_z S^{2n+1}$ is the natural connection $\mathfrak{A}$ on the principal $U(1)$-bundle $\pi : S^{2n+1} \to \mathbb{CP}^n$ associated to the standard metric on S^{2n+1}.

Recall that the canonical metric on $\mathbb{CP}^n$ is defined to be the one for which the Hopf fibration π is an isometry. Then obviously

$$d\pi : H_z \longrightarrow T_{\pi(z)} \mathbb{CP}^n$$

is also an isometry of linear spaces.

The above discussion together with the calculations adapted from the proof of proposition 36 (all we have to do is to change the Euclidean product by the Hermitian one: $\langle X, Y \rangle = \sum_{i=0}^n x_1 \bar{y}_i$) gives us the proof of the following proposition. Denote as usual $|z|^2 = z \cdot \bar{z}$ for $z \in \mathbb{C}$.

Proposition 37 *Let the system* $(T^*S^{2n+1}, \omega_{can}, H_{(c4)})$ *be given by the Hamiltonian*

$$H_{(c4)} = \|p\|^2 + 4\langle \beta q, \beta q\rangle - |\langle \beta q, q\rangle|^2 \tag{3.70}$$

Let $\vartheta : S^{2n+1} \to \mathcal{CP}^n \cong \mathbb{CP}^n$ *be the Hopf fibration given by the formula* $\vartheta(Z) = (I - 2Z^*Z)\cdot J$. *Then the mapping* $d^*\vartheta : T^*\mathcal{CP}^n \to T^*S^{2n+1}$ *maps the solutions of the system* $(T^*\mathcal{CP}^n, \omega_{can}, H)$ *into the solutions of* $(T^*S^{2n+1}, \omega_{can}, H_{(c4)})$.

Equivalently, let $\gamma(t) : I \to \mathcal{CP}^n \cong \mathbb{CP}^n$ *be a solution of the variational problem given by the Lagrangian*

$$L(k, \dot{k}) = \|k^{-1}\dot{k}\|^2 + \mathcal{K}(Ad_k(\beta), \widetilde{\beta}) \, ,$$

and let $Z_0 \in \vartheta^{-1}(\gamma(0))$ *be fixed. Then the unique lifting* $\widetilde{\gamma}(t) : I \to S^{2n+1}$ *given by the connection* $\mathfrak{A}$ *and the initial point* Z_0 *is a solution of the variational problem on* S^{2n+1} *given by the Lagrangian*

$$L_{(c4)}(Z, \dot{Z}) = \|\dot{Z}\|^2 - 4\langle \beta Z, \beta Z\rangle + |\langle \beta Z, Z\rangle|^2 \, .$$

□

Corollary 3 *The Hamiltonian system* $(T^*S^{2n+1}, \omega_{can}, H_{(c4)})$ *is completely integrable*

The integrability of the above system is clear from the fact that this system is a special case of the one described in proposition 36. Nevertheless, we give a short independent proof.

The $U(1)$-action on S^{2n+1} lifts naturally to the symplectic $U(1)$-action on T^*S^{2n+1}. In addition, the Hamiltonian $H_{(c4)}$ is invariant with respect to this action. Let

$$\mu : T^*S^{2n+1} \longrightarrow \mathfrak{u}(1) = i\mathbb{R} \cong \mathbb{R}$$

be the corresponding moment map and $0 \in \mathbb{R}$. Then the system $(T^*\mathcal{CP}^n, \omega_{can}, H)$ is equivalent to the system on the symplectic quotient $(\mu^{-1}(0)/U(1), \widetilde{\omega}, \widetilde{H}_{(c4)})$. This system is integrable and therefore, by lemma 1, so is the system $(T^*S^{(n+1)}, \omega_{can}, H_{(4)})$.

Lastly we address our Hamiltonian system (T^*M, ω_{can}, H) where M is the the quaternionic projective space $\mathbb{HP}^n$. Recall that $\mathbb{HP}^n \cong Sp(n)/(Sp(1) \times Sp(n))$. The realization of $\mathbb{HP}^n$ as a subspace of $Sp(n)$ will be denoted by $\mathcal{HP}^n$. Let $W \in \mathbb{H}^n$ be given as a vector of quaternions $W = (w_1, \ldots, w_n)$, $w_i \in \mathbb{H}$, and let $W^* = (\bar{W^T})$, where $\bar{W}$ denotes the quaternionic conjugation.

Adapting in a straightforward way the proof of lemma 13, we prove the following one.

Lemma 14 *Elements $f \in \mathcal{HP}^n$ are the symplectic linear maps given by the matrices of quaternions*

$$f = (I - 2W^*W) \cdot J \ ,$$

for some $W \in S^{4n+3}$. The mapping

$$\chi : S^{4n+3} \longrightarrow \mathcal{HP}^n \cong \mathbb{HP}^n$$

given by $\chi(W) = (I - 2W^*W)J$ is the Hopf fibration with the fibre $SU(2) \cong S^3$.

□

The natural connection $\mathfrak{A}_{\mathbb{H}}$ on the principal $SU(2)$-bundle $\pi : S^{4n+3} \to \mathbb{HP}^n$ is induced by the orthogonal decomposition

$$\mathbb{H}^n = \mathbb{R} \cdot q \oplus (\mathbb{R} \cdot iq \oplus \mathbb{R} \cdot jq \oplus \mathbb{R} \cdot kq) \oplus H_q$$

at every point $q \in S^{4n+3} \subset \mathbb{H}^n$.

The isomorphism $\mathbb{H} \to \mathbb{C} \oplus j \cdot \mathbb{C}$ induces the standard embedding of $Sp(n)$ into $U(2n)$ defined by

$$f \mapsto \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} ,$$

where A, B are $n \times n$ matrices. In this representation the Killing form on $\mathfrak{sp}(n)$ is given by $\mathcal{K}(f_1, f_2) = Tr(f_1 \cdot f_2)$. Therefore, the calculations from proposition 36 can be used again to prove the following .

Proposition 38 *Let the Hamiltonian of the system $(T^*S^{4n+3}, \omega_{can}, H_{(q4)})$ be given by*

$$H_{(q4)} = \|p\|^2 + 4\langle \beta q, \beta q\rangle - |\langle \beta q, q\rangle|^2 \ , \tag{3.71}$$

where $\beta \in Sp(n) \subset U(2n)$. Let $\chi : S^{4n+3} \to \mathcal{HP}^n$ be the Hopf fibration with the fibre $SU(2)$. Then

$$d^*\chi : T^*\mathcal{HP}^n \to T^*S^{4n+3}$$

maps the solutions on $(T^*\mathcal{HP}^n, \omega_{can}, H)$ to the solutions of $(T^*S^{4n+3}, \omega_{can}, H_{(q4)})$.

Let a solution of $(T^*\mathcal{HP}^n, \omega_{can}, H)$ be given as a path $\gamma(t) : I \to \mathcal{HP}^n \cong \mathbb{HP}^n$. Then its lifting to S^{4n+3} horizontal with respect to the connection $\mathfrak{A}_{\mathbb{H}}$ is a solution of the system $(T^*S^{4n+3}, \omega_{can} H_{(q4)})$.

The system $(T^*S^{4n+3}, \omega_{can}, H_{(q4)})$ belongs to the class described in proposition 37 and is therefore integrable in the Liouville sense by corollary 3.

□

The respective $U(1)$ and $SU(2)$ invariance of the Hamiltonian systems described in propositions 37 and 38 allows us to construct integrable systems on certain spaces which have the spheres as the universal covering spaces.

Recall that the generalised lens space is defined to be the quotient space $S^{2n+1}/\mathbb{Z}_k$, where the cyclic group $\mathbb{Z}_k$ action is given by

$$p \cdot (Z_0, Z_1, \ldots, Z_n) = (e^{p(2\pi i/k)} Z_0,\ e^{p(2\pi i k_1/k)} Z_1,\ \ldots, e^{p(2\pi i k_n/k)} Z_n)\ ,$$

where the integers $0 \leq k_i < k$ are fixed. The quotient of S^{2n+1} with respect to such action is denoted by $\mathcal{L}(k;\ k_1, \ldots, k_n)$. We can lift this action on a symplectic action of $\mathbb{Z}_k$ on T^*S^{2n+1}. The quotient space is the cotangent bundle $T^*\mathcal{L}(k:\ k_1, \ldots, k_n)$. Since the Hamiltonian $H_{(c4)}$ is invariant with respect to the $U(1)$-action, it is also invariant with respect to the actions of the subgroups of $U(1)$. Therefore, we get an induced Hamiltonian $H_{(L4)}$ on $T^*\mathcal{L}(k;\ k_1, \ldots k_n)$. The following is a consequence Corollary 3.

Corollary 4 *The Hamiltonian system $(T^*\mathcal{L}(k;\ k_1, \ldots k_n), \omega_{can}, H_{(L4)})$ is a completely integrable system for every generalised lens space $\mathcal{L}(k;\ k_1, \ldots k_n)$.*

□

The integrable systems on quotients of S^{4n+3} by finite subgroups of $SU(2)$ can be produced in the same way as the ones on the lens spaces. This might be of some interest since it gives us some integrable systems on spaces with nontrivial fundamental groups, namely the cyclic groups $\mathbb{Z}_k$ as well as the other finite subgroups of $SU(2)$, e.g. the icosahedral group.

3.5.3

Particle in the magnetic field

We have mentioned above that the integrable system $(T^*\mathcal{CP}^n, \omega_{can}, H)$ is the symplectic quotient of the system $(T^*S^{(2n+1)}, \omega_{can}, H_{(c4)})$. More precisely,

$$(T^*\mathcal{CP}^n\ ,\ \omega_{can}) = (\mu^{-1}(0)/U(1)\ ,\ \omega_{can})\ ,$$

where $\mu : T^*S^{(2n+1)} \to i \cdot \mathbb{R}$ is the moment map of the cotangent lifting of the usual $U(1)$-action on $S^{(2n+1)}$ given by

$$u \cdot (z_0, z_1, \ldots, z_n) = (uz_0, uz_1, \ldots, uz_n)\ ,$$

and the Hamiltonian H comes from the Hamiltonian $H_{(c4)}$.

Let now $\gamma \in i \cdot \mathbb{R} = \mathfrak{u}(1)$ be a regular value of μ different from the zero. The manifold $\mu^{-1}(\gamma)/U(1)$ is still diffeomorphic to the space $T^*\mathcal{CP}^n$, but the induced symplectic form ω_{ind} is not the canonical one this time.

Lemma 15 *Let $\omega_{(FS)}$ denote the Fubini-Study form on the space $\mathbb{CP}^n$, and let $\omega^*_{(FS)}$ be its lifting on the cotangent bundle $T^*\mathbb{CP}^n$. Then we have*

$$(\mu^{(-1)}(\gamma)/U(1)\ ,\ \omega_{(ind)}) \cong (T^*\mathbb{CP}^n\ ,\ \omega_{can} + \gamma\omega^*_{(FS)})\ .$$

Proof: Recall that the mechanical connection $\alpha : TM \longrightarrow \mathfrak{g}$ of the G-action on M is given by the formula

$$\alpha(x, v) = \rho^{-1}(x)(\mu(Leg(x, v)))\ ,$$

where $\rho : \mathfrak{g}^* \to \mathfrak{g}$ is defined by

$$\langle \rho(x)\xi\ ,\ \eta\rangle_{\mathfrak{g}} = \langle \xi_M\ ,\ \eta_M\rangle_x\ .$$

The mapping $\mu : T^*M \to \mathfrak{g}^*$ is the moment map of the G-action lifted on T^*M, and *Leg* is the Legendre transformation. Here $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and $\langle\cdot,\cdot\rangle_x$ are the dual pairings of $\mathfrak{g}$ with $\mathfrak{g}^*$, and of T_xM with T^*_xM respectively. The symbols ξ_M, η_M denote the infinitesimal actions of ξ and η on M. Let the one-form α_γ on M be defined by

$$\langle \alpha_\gamma(x)\ ,\ v\rangle_x = \langle \gamma\ ,\ \alpha(x, v)\rangle_{\mathfrak{g}}$$

for some $\gamma \in \mathfrak{g}^*$. Denote by G_γ the stabiliser of γ and by β_γ the form on M/G_γ, such that $\pi^*\beta_\gamma = d\alpha_\gamma$ where $\pi : M \to M/G_\gamma$ is the natural projection. It is proved in Chapter 3 of [Ma], that

$$(\mu^{-1}(\gamma)/G_\gamma\ ,\ \omega_{ind}) = (\mu^{-1}(\gamma)/G_\gamma\ ,\ \widetilde{\omega}_{can} + \widetilde{\beta^*_\gamma})\ .$$

The forms $\widetilde{\omega}_{can}$, and $\widetilde{\beta^*_\gamma}$ are the restrictions of ω_{can}, β^*_γ from $T^*(M/G_\gamma)$ to the subspace $\mu^{-1}(\gamma)/G_\gamma$. Note that the spaces $\mu^{-1}(\gamma)/G_\gamma$ and T^*M/G_γ coincide whenever G is Abelian.

Let now $\mu : T^*S^{(2n+1)} \to i\mathbb{R}$ be the moment map of our $U(1)$-action. It is given by the formula

$$\langle \mu(z, p)\ ,\ \xi\rangle = \langle p\ ,\ \xi_{S^{2n+1}}(z)\rangle_{\mathbb{R}}\ ,$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ denotes the real part the Hermitian product on $\mathbb{C}^{(n+1)}$. Since $\xi_{S^{(2n+1)}}(z) = \xi \cdot iz$, we get immediately

$$\mu(z, p) = \langle p\ ,\ iz\rangle_{\mathbb{R}}\ .$$

(This shows that $\mu^{-1}(0) = H_z^*$, where H_z is the orthogonal complement of $iz \in T_z S^{(2n+1)} \subset \mathbb{C}^{n+1}$) Since $\rho(z) \equiv 1$ this gives

$$\alpha(z,v) = \langle v\ , iz\rangle_{\mathbb{R}}\ ,$$

and

$$\alpha_\gamma(z,v) = \gamma \cdot \langle v\ ,\ iz\rangle_{\mathbb{R}}\ .$$

Denoting $z_j = x_j + iy_j$, we then get

$$\alpha_\gamma(z) = \gamma \cdot \sum_{j=0}^{n} -y_j dx_j + ix_j dy_j\ ,$$

and finally

$$d\alpha_\gamma(z) = \gamma(\frac{i-1}{2}) \cdot \sum_{j=0}^{n} dz \wedge d\bar{z}\ .$$

The space $\mathbb{CP}^n$ can be thought of as the symplectic quotient of $\mathbb{C}^{(n+1)}$ with respect to the obvious $U(1)$-action $u \cdot (z_0, z_1, \ldots, z_n) = (uz_0, uz_1, \ldots uz_n)$. The corresponding moment map has the expression

$$\Phi(z_0, z_i, \ldots, z_n) = |z_0|^2 + |z_1|^2 + \ldots + |z_n|^2\ ,$$

hence $\Phi^{-1}(1)/U(1) \cong S^{(2n-1)}/S^1 = \mathbb{CP}^n$. The Fubini-Study form on $\mathbb{CP}^n$ comes from the form

$$\partial\bar{\partial} \log \|z\|^2 = \partial \left(\frac{\sum_{j=0}^{n} z d\bar{z} + \bar{z} dz}{\|z\|^2} \right)\ .$$

Since on the sphere $S^{(2n-1)}$ we have $d\|z\|^2 = 0$, the restrictions of the forms $d\alpha_\gamma$ and $\partial\bar{\partial}\log\|z\|^2$ on the sphere coincide up to a constant factor and therefore descent to essentially the same form $\omega_{(FS)}$ on the space $\mathbb{CP}^n$. □

Denote by $H_{(m)} : T^*\mathcal{CP}^n \to \mathbb{R}$ the Hamiltonian induced from $H_{(c4)}$ via the symplectic quotient $\mu^{-1}(\gamma)/U(1) \cong T^*\mathcal{CP}^n$. It is easily seen from the discussion in the first part that $H_{(m)}$ comes from the $U(1)$-invariant function $F(z,p) : \mu^{-1}(0) \to \mathbb{R}$ defined by

$$F(z,p) = H_{(c4)}(z, p - \alpha_\gamma(z))\ .$$

(The explicit proof of the above relation can be found e.g. in [Ma], Chapter 3.) A short calculation then shows that

$$H_{(m)} = H + \|\gamma\|^2, \tag{3.72}$$

where $H : T^*\mathcal{CP}^n \to \mathbb{R}$ is our usual Hamiltonian on $T^*\mathcal{CP}^n$. So the Hamiltonians $H_{(m)}$ and $H_{(c4)}$ are essentially the same, since they differ only by the constant $\|\gamma\|^2$. The additional magnetic force, affecting the particle moving on $\mathbb{CP}^n$, comes from the deformation of the canonical symplectic form ω_{can} on $T^*\mathbb{CP}^n$ by the multiple of the Fubini-Study form $\gamma\omega_{(FS)}$ on $\mathbb{CP}^n$. The integrability of the system $(T^*S^{(2n+1)}, \omega_{can}, H_{(c4)})$ gives us the proof of the following proposition.

Proposition 39 *The system $(T^*\mathbb{CP}^n, \omega_{can}+\gamma\omega^*_{(FS)}, H_{(m)})$ with the Hamiltonian $H_{(m)}$ defined by 3.72 describing the motion of a particle under the influence of the force potential $V_\beta = \mathcal{K}(Ad_k(\beta), \widetilde{\beta})$ and the magnetic force generated by the magnetic term $\gamma\omega_{(FS)}$ on $\mathbb{CP}^n$, is a completely integrable system.*

□

In particular, taking $\beta = 0$ gives the integrability of the motion in the field of the magnetic force alone. In the case when $n = 2$ our system describes the motion of a particle in $\mathbb{R}^3$ in the field of forces generated by a quadratic potential and a magnetic monopole situated in $0 \in \mathbb{R}^3$. The particle is in addition confined to the sphere $\mathbb{CP}^1 \cong S^2 \subset \mathbb{R}^3$.

Since the Hamiltonians $H_{(m)}$ and H differ only by an additive constant, the Lorentz force does not depend on the position, but only on the momentum, the relevant contribution being determined by the form $\omega_{(FS)}$ on $\mathbb{CP}^1$. It can be easily seen that the strength B of the monopole is given by the expression $B = \gamma\frac{x}{\|x\|^3}$ where $x \in \mathbb{R}^3$. The spherical symmetry of this monopole explains the independence on the position of the Lorentz force.

Recall proposition 35 describing the spherical pendulum. It was obtained as the symplectic quotient $\mu^{-1}(0)/U(1)_\beta$ of the system $(T^*SO(3), \omega_{can}, H)$, where $H(q,p) = \|p\|^2 + \mathcal{K}(Ad(\beta), \beta)$. Taking a non-zero $\gamma \in \mathfrak{so}(3)$ lying in the same torus sub-algebra as β and proceeding exactly in the same way as above, gives us the following corollary.

Corollary 5 *The system describing the spherical pendulum moving in the field of a magnetic monopole placed in the centre of the sphere is a completely integrable system.*

□

Particle in the Yang-Mills field

The configuration space of a particle moving on a manifold M in the presence of a Yang-Mills field is a principal bundle $P \to M$ with the structure group $SU(2)$. The group sitting above each point of M parametrises the possible states of the particle, i.e. represents its internal structure. The appropriate phase space is then $(T^*P\ ,\ \omega_{can})$. Having the natural $SU(2)$-action it makes sense to reduce T^*P symplectically. Let $\mu : T^*P \to \mathfrak{su}^*(2)$ be the moment map of the $SU(2)$-action lifted on T^*P, and let $\gamma \in \mathfrak{su}(2)^*$ be a regular value. Let $N = \mu^{-1}(\gamma)/U(1)$ be the symplectic quotient space, $U(1)$ being the centraliser of γ. It is easily seen that N has the structure of a S^2-fibre bundle $N \to T^*M$. The sphere S^2 appears here because it is the coadjoint orbit $\mathcal{O}_\gamma \subset \mathfrak{su}(2)^*$ of γ. We will describe a concrete example of this situation in more detail below. By analogy with the situation in the electro-magnetic theory, we think of $\mathcal{O}_\gamma$ as of the generalised charge of the particle in the Yang-Mills field. In the electromagnetism the structure group is $U(1)$ and so $\mathcal{O}_\gamma = \gamma$.

We shall now treat the system $(T^*S^{(4n+3)}, \omega_{can}, H_{(q4)})$ in an analogous manner as we treated the system with the Hamiltonian $H_{(c4)}$ above. There is a natural $SU(2)$-action on $S^{(4n+3)}$ given by

$$q \cdot (w_0, w_1, \ldots, w_n) = (qw_0, qw_1, \ldots, qw_n)\ ,$$

where the elements of $SU(2)$ are identified with the unit quaternions $q \in S^3 \subset \mathbb{H}$. This action makes $S^{(4n+3)} \to \mathbb{HP}^n$ into a $SU(2)$-principal bundle. The infinitesimal action of $\xi \in \mathfrak{su}(2) \cong Im(\mathbb{H})$ is then the vector field $\xi_S(w) = (\xi w_0, \xi w_1, \ldots, \xi w_n)$. In order to compute the moment map $\mu : T^*S^{(4n+3)} \to \mathfrak{su}^*(2)$ it is convenient to represent the quaternions as the complex matrices of the form

$$q = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} .$$

The usual Euclidean product on $\mathbb{H}$ is then $\langle q_1, q_2 \rangle = Tr(q_1 \cdot q_2^*)$. The defining equation

$$\langle \mu(w,p)\ ,\ \xi \rangle_{\mathfrak{su}(2)} = \langle p\ ,\ \xi_S(w) \rangle_{T_w S^{(4n+1)}}$$

for the moment map of the $SU(2)$-action lifted on $T^*S^{(4n+1)}$ can be written as

$$Tr(\mu(w,p) \cdot \xi) = Tr((\sum_{j=0}^{n} p_j w_j^*) \cdot \xi^*) = Tr((\sum_{j=0}^{n} w_j p_j^*) \cdot \xi)\ ,$$

which yields the expression

$$\mu(w,p) = \sum_{j=0}^{n} w_j p_j^* . \tag{3.73}$$

Putting $w_j = \det(w_j) \cdot \widetilde{w_j}$, this can also be written as

$$\mu(w,p) = \sum_{j=0}^{n} \det(w_j)^2 \cdot Ad_{\widetilde{w_j}} \widetilde{p_j} \; ,$$

for some $\widetilde{p_j} \in \mathfrak{su}(2)$, which confirms, that μ takes values in $\mathfrak{su}(2)$. From 3.73 we see that $\mu^{-1}(0)$ is the sub-bundle of $T^*S^{(4n+3)}$ whose fibre at the point $w \in S^{(4n+3)}$ is the dual space of the subspace $H_w \subset T_w S^{(4n+3)}$ orthogonal to the $span(iw, jw, kw)$. Therefore $\mu^{-1}(\gamma)$ is the bundle of affine subspaces $(\alpha_\gamma(w) + H_w^*) \subset T_w^* S^{(4n+3)}$.

Denote the symplectic quotient $\mu^{-1}(\gamma)/U(1)$ by $\mathcal{N}$. Then $\mathcal{N} \to T^*\mathbb{HP}^n$ is the fibre bundle having the 2-sphere S^2 as the fibre. This is so, because the group $SU(2)$ acts only on the vertical part of the principal bundle $S^{(4n+3)} \to \mathbb{HP}^n$. When lifted on $T^*S^{(4n+3)}$ the group $SU(2)$ acts on $T^*SU(2)$ and the symplectic quotient of this action is S^2. Alternatively, $\mathcal{N}$ can be viewed as a sub-bundle of the cotangent bundle $T^*\mathbb{CP}^{(2n+1)}$, since $S^{(4n+3)}/U(1) = \mathbb{CP}^{(2n+1)}$. For every n there is a natural S^2-fibration $\mathbb{CP}^{(2n+1)} \to \mathbb{HP}^n$ given by the mapping of lines $z \cdot \mathbb{C} \to (z + j \cdot z) \cdot \mathbb{H}$. The mechanical connection induces the direct sum decomposition

$$T_a^*\mathbb{CP}^{(2n+1)} = T_a^* S^2 \oplus T_a^*\mathbb{HP}^n \; .$$

at every point $a \in \mathbb{CP}^{(2n+1)}$. The vertical part $T_a^* S^2$ is determined by the fibration, while the horizontal part $T_a^*\mathbb{HP}^n$ is fixed as the dual of the kernel of the mechanical connection α. Then $\mathcal{N} \subset T^*\mathbb{CP}^{(2n+1)}$ is the sub-bundle with the fibre $T_a^*\mathbb{HP}^n$ over the point $a \in \mathbb{CP}^{(2n+1)}$.

Since we have $\rho(w) \equiv 1$, the mechanical connection on the $SU(2)$-principal bundle $S^{(4n+3)} \to \mathbb{HP}^n$ has the form

$$\alpha(w,v) = \sum_{j=0}^{n} w_j v_j^* \; .$$

For an arbitrary $\gamma \in \mathfrak{su}(2)^*$ the one-form α_γ acts by the rule

$$\langle \alpha_\gamma(w) \; , \; v \rangle_{T_w S^{(4n+3)}} = Tr((\sum_{j=0}^{n} w_j v_j^*) \cdot \gamma) \; .$$

Let for the sake of simplicity $\gamma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and denote $w_j = (z_j, z_{j+n+1})$ for every j. Then we can write α_γ in the form

$$\alpha_\gamma = i \cdot \sum_{j=0}^{n} (z_j d\bar{z}_j - \bar{z}_j dz_j + z_{j+n+1} d\bar{z}_{j+n+1} - \bar{z}_{j+n+1} dz_{j+n+1}) \; ,$$

and hence

$$d\alpha_\gamma = 2i \cdot (\sum_{j=0}^{n+1} dz_j \wedge d\bar{z}_j) \; . \qquad (3.74)$$

From the expression 3.74 we see that the 2-form $d\alpha_\gamma$ is horizontal on the bundle $T^*S^{(4n+3)}$, i.e. $(d\alpha_\gamma)_a(v_1, v_2) = 0$ if v_1 or v_2 is tangent to $T^*_a S^{(4n+3)}$. That remains to be true after taking the symplectic quotient. More precisely, $d\alpha_\gamma$ is $U(1)$-invariant, so its restriction on $\mu^{-1}(\gamma)$ induces the form β_γ on $\mathcal{N} = \mu^{-1}(\gamma)/U(1)$. Then obviously $(\beta_\gamma)_a(v_1, v_2) = 0$ if v_1 or v_2 lies in $T_a(T^*\mathbb{HP}^n) \subset T_a\mathcal{N}$. The same argument as in the the proof of lemma 15 shows that $d\alpha_\gamma$ descends to the lifting of the Fubini-Study form $\omega_{(FS)}$ from $\mathbb{CP}^{(2n+1)}$ to $\mathcal{N} \to \mathbb{CP}^{(2n+1)}$.

The above proves the following lemma.

Lemma 16 *Let $\widetilde{\omega}_{can}$ denote the restriction of the canonical symplectic form on the cotangent bundle $T^*\mathbb{CP}^{(2n+1)}$ to the sub-bundle $\mathcal{N}$, and let $\widetilde{\omega}_{(FS)}$ be the lifting of the Fubini-Study form from $\mathbb{CP}^{(2n+1)}$ to the total space of the bundle $\mathcal{N} \to \mathbb{CP}^{(2n+1)}$. Then the symplectic quotient of $T^*S^{(4n+3)}$ has the form*

$$(\mu^{-1}(\gamma)/U(1)\ ,\ \omega_{ind}) \cong (\mathcal{N}\ ,\ \widetilde{\omega}_{can} + \widetilde{\omega}_{(FS)})\ .$$

□

While it is easier to describe the induced symplectic form if we think of $\mathcal{N}$ as a sub-bundle of $T^*\mathbb{CP}^{(2n+1)}$, the expression of the induced Hamiltonian $H_{(YM)}$ will be more suggestive for us if we place it into $\mathcal{N}$ viewed as the S^2-bundle over $T^*\mathbb{HP}^n$, since this will enable us to compare it with the Hamiltonian H of the previously discussed system $(T^*\mathcal{HP}^n, \omega_{can}, H)$.

Let M be a symplectic manifold equipped with a Hamiltonian G-action, let $\gamma \in \mathfrak{g}^*$ and let $\mathcal{O}_\gamma$ be the coadjoint orbit of γ. It is well known that the spaces $\mu^{-1}(\gamma)/G_\gamma$ and $\mu^{-1}(\mathcal{O}_\gamma)/G$ are symplectically the same. Let $M = T^*Q$ and let the G-action come from an action on Q. It is then easily seen that the map $\mu^{-1}(\mathcal{O}_\gamma)/G \to T^*(Q/G)$ is a fibre bundle having the coadjoint orbit $\mathcal{O}_\gamma$ as the fibre. In our case, we have the already mentioned S^2-bundle $\mathcal{N} \to T^*\mathcal{HP}^n$. Similarly as in the previous paragraph we conclude that the Hamiltonian $H_{(YM)}$ comes from the $SU(2)$ invariant function F defined on $\mu^{-1}(\mathcal{O}_\gamma)$. In local coordinates (w, p, q) where (w, p) are local coordinates on $T^*\mathcal{HP}^n$ and q is a coordinate on $\mathcal{O}_\gamma = S^2$ we have

$$F(w, p, q) = H_{(q4)}(w, p - \alpha_q(w), q)\ .$$

Again a short calculation shows that after quotienting by $SU(2)$ we get

$$H_{(YM)}(w, p, q) = H(w, p) + \|\alpha_q\|^2, \tag{3.75}$$

where H is the Hamiltonian of the system $(T^*\mathcal{HP}^n, \omega_{can}, H)$. Since $\|\alpha_q\|^2 \equiv \|\alpha_\gamma\|^2$, we finally get

$$H_{(YM)}(w, p, q) = H(w, p) + \|\alpha_\gamma\|^2\ .$$

The above discussion and the already established integrability of the system $(T^*S^{(4n+3)}, \omega_{can}, H)$ proves the following proposition.

Proposition 40 *The system $(\mathcal{N}, \widetilde{\omega}_{can}, H_{(YM)})$ where $H_{(YM)}$ is defined by 3.75 describing the motion of a particle on $\mathbb{HP}^n$ in the field of the Yang-Mills force determined by the form $\widetilde{\omega}_{(FS)}$, and in the potential field given by the potential $V_\beta(k) = \mathcal{K}(Ad_k\beta, \widetilde{\beta})$ is a completely integrable system. In particular, taking $\beta = 0$ this gives the integrability of the system where the particle moves in the Yang-Mills field alone.*

□

In the case where $n = 1$ we get the motion on the sphere S^4. The space $\mathcal{N}$ is a sub-bundle of $T^*\mathbb{CP}^3$. The S^2 fibre bundle $\mathbb{CP}^3 \to S^4$ is the twistor space of S^4 and the Yang-Mills field is easily seen to be given by the basic instanton on S^4 as described in [At].

3.6 Harmonic maps and loop groups

In this section we collect a few brief remarks concerning the relationship between Nahm's equations and the harmonic maps. The systems we were considering in this chapter were all derived from Nahm's equations for the functions taking values in some semi-simple Lie algebra. Obviously, Nahm's equations make sense for the functions taking values in any Lie algebra. One possibility to place Nahm's equations in a different framework is to consider the system

$$\begin{gathered} \dot{T}_i + [T_0, T_i] + \tfrac{1}{2} \sum \varepsilon_{i,j,k} \ [T_j, T_k] = 0 \\ T_0, T_i : I \longrightarrow Lie(\Omega G) \end{gathered} \tag{3.76}$$

where ΩG denotes the loop group over a semi-simple Lie group G, i.e. the group of maps $S^1 \to G$. This gives rise to perturbations of the harmonic maps. For a suitable choice of certain parameter we can actually obtain the harmonic maps themselves. Discussion of this topic would deserve a chapter of its own, but time allows us only to take a short and sketchy glimpse of it.

Our intention is to associate to system 3.76 the Lagrangian functional in the same way as we did in subsection 3.1.2.

Instead of the groups ΩG it is more interesting to consider the semi-direct product $\widetilde{\Omega} G = \mathbb{T} \triangleleft \Omega G$ where $\mathbb{T}$ denotes the circle group which acts on the elements of ΩG by rotating them, i.e.

$$u \cdot g(s) = g(s+u) \ ,$$

where $u \in \mathbb{T}$, and $g(s) : S^1 \to G$. This group is used by Garland and Murray in [G-M 1], and [G-M 2], where they interpret the periodic instantons as the monopoles having $\widetilde{\Omega} G$ as the structure group. The Lie algebra $Lie(\widetilde{\Omega} G)$ is clearly the extension $Lie(\Omega G) \oplus \frac{\partial}{\partial s} \cdot i\mathbb{R}$, where the generator $\frac{\partial}{\partial s}$ is the infinitesimal rotation.

We will denote the elements of $\widetilde{\Omega} G$ by $\hat{g} = (g(s), u)$, where u is the rotation, and the elements of $Lie(\widetilde{\Omega} G)$ by $\hat{\gamma} = (\gamma(s), c)$. The multiplication law for the semi-direct products gives

$$\hat{g} \cdot \hat{f} = (g(s), e^u) \cdot (f(s), e^v) = (g(s)f(s+u) \ , \ e^{(u+v)}) \ .$$

From this we get the expression

$$Ad_{\hat{g}}(\hat{f}) = (g(s)f(s+u)g^{-1}(s+v) \ , \ e^v) \ .$$

Let $\hat{f}(t) : I \to \widetilde{\Omega} G$ be a path starting at $e \in \widetilde{\Omega} G$ and having $\hat{\gamma}$ as the tangent there. Taking the derivative $\frac{d}{dt}|_{t=0} Ad_{\hat{g}}(\hat{f}(t))$ gives the following expression for the adjoint action of $\widetilde{\Omega} G$ on $Lie(\widetilde{\Omega} G)$

$$Ad_{\hat{g}}(\hat{\gamma}) = (Ad_g(\gamma(s+u)) - c \cdot g_s g^{-1} \ , \ c) \ , \tag{3.77}$$

where $g_s = \frac{d}{ds} g$. Deriving 3.77 along the path $\hat{g} : I \to \widetilde{\Omega G}$ with $(\frac{d}{dt}(\hat{g}))(0) = \hat{\beta} = (\beta(s)\ , b)$, we get the formula for the Lie bracket in $Lie(\widetilde{\Omega G})$

$$[\hat{\beta}, \hat{\gamma}] = ([\beta(s), \gamma(s)] + b \cdot \gamma_s - c \cdot \beta_s \quad , \quad 0)\,. \tag{3.78}$$

The non-degenerate Ad-invariant Killing form on $Lie(\widetilde{\Omega G})$ is given by

$$\widetilde{\mathcal{K}}(\hat{\beta}\ ,\ \hat{\gamma}) = \int_{S^1} \mathcal{K}(\beta\ ,\ \gamma)\,. \tag{3.79}$$

In order to carry out Donaldson's rewriting of Nahm's equations in the variational form, one needs the complexifications of the relevant Lie algebra and of its Lie group. While every Lie algebra clearly has a complexification, finding one for a Lie group is not always easy or even possible. In our case the necessary complexifiations do exist. For the Lie algebra we have

$$Lie(\widetilde{\Omega G})^{\mathbb{C}} = \{\gamma^{\mathbb{C}} : \mathbb{C}^* \to \mathfrak{g}^{\mathbb{C}}\} \oplus \frac{d}{dz} \cdot \mathbb{C}\,,$$

and for the group

$$\widetilde{\Omega G}^{\mathbb{C}} = \mathbb{C}^* \triangleleft \{g : \mathbb{C}^* \to G^{\mathbb{C}}\}\,,$$

where $\mathbb{C}^*$ acts in the obvious way by $\rho e^{i\xi} \cdot g^{\mathbb{C}}(re^{is}) = g^{\mathbb{C}}((\rho r)e^{i(\xi+s)})$. The real structure $\tau : Lie(\widetilde{\Omega G})^{\mathbb{C}} \to Lie(\widetilde{\Omega G})^{\mathbb{C}}$ whose real form is $\widetilde{\Omega G}$ is given by

$$\tau(\hat{g}^{\mathbb{C}}) = \tau(g^{\mathbb{C}}(z)\ ,\ a) = (g^{\mathbb{C}}(\bar{z}^{-1})\ ,\ \bar{z}^{-1})\,. \tag{3.80}$$

The adjoint action of $\widetilde{\Omega G}^{\mathbb{C}}$ on $Lie(\widetilde{\Omega G})^{\mathbb{C}}$ and the bracket on $Lie(\widetilde{\Omega G})^{\mathbb{C}}$ have the same expression as 3.77 and 3.78 respectively.

The key ingredient of the variational interpretation of Nahm's equations is proposition 21 on page 84 and the rewriting two of the equations in Nahm's system in the form of a Lax equation for the variables in the complexification of the algebra. An inspection of the proof of the proposition 21 shows that all we need is the complexification of the Lie group involved and a non-degenerate Killing form on it. Since the Killing form on $Lie(\widetilde{\Omega G})^{\mathbb{C}}$ is given by

$$\widetilde{\mathcal{K}}(\hat{\beta}\ ,\ \hat{\gamma}) = \int_{\mathbb{C}^*} \mathcal{K}(\hat{\beta}\ ,\ \hat{\gamma})\,, \tag{3.81}$$

we must restrict $Lie(\widetilde{\Omega G})^{\mathbb{C}}$ to a class of functions $\beta : \mathbb{C}^* \to \mathfrak{g}^{\mathbb{C}}$, such that the integrals of form 3.81 will converge. Having taken care of that, we get the following corollary of proposition 22.

Corollary 6 *The solutions of Nahm's system for the loop algebra valued functions*

$$\dot{\hat{T}}_i + \frac{1}{2} \sum \varepsilon_{i,j,k} \; [\hat{T}_j, \hat{T}_k] = 0$$

$$\hat{T}_0, \hat{T}_i : I \longrightarrow Lie(\widetilde{\Omega} G)$$

are in one-to-one correspondence with the solutions of the variational problem on the space $\Omega\mathcal{H} = \widetilde{\Omega} G^{\mathbb{C}} / \widetilde{\Omega} G$ *given by the Lagrangian*

$$\mathcal{L}(\hat{h}) = \int_{\mathbb{R}} \|\hat{h}_t\|^2_{\Omega\mathcal{H}} + \widetilde{\mathcal{K}}(Ad_{\hat{h}}(\hat{\beta}) \; , \; \tilde{\hat{\beta}}) \; .$$

We note that the space $\Omega(\mathcal{H})$ is not simply the space of loops on $\mathcal{H} = G^{\mathbb{C}}/G$, and describing it would take some time. Instead, we are going to concentrate on a more readily manageable situation where the configuration space of the variational problem will be the loop group $\widetilde{\Omega} G$. We are going to proceed in the analogous way that we took in subsection 3.2.3, considering Nahm's equations for the functions with values in the complex Lie algebra $Lie(\widetilde{\Omega} G)^{\mathbb{C}}$. Replacing $Lie(\widetilde{\Omega} G)$ by $Lie(\widetilde{\Omega} G)^{\mathbb{C}}$ in the above corollary yields the variational problem on the space $\widetilde{\Omega} G^{\mathbb{C}} \cong (\widetilde{\Omega} G^{\mathbb{C}} \times \widetilde{\Omega} G^{\mathbb{C}}) / \widetilde{\Omega} G^{\mathbb{C}}_r$ where $\widetilde{\Omega} G^{\mathbb{C}}_r = \{(\hat{g}, \hat{g}) \; ; \; \hat{g} \in \widetilde{\Omega} G^{\mathbb{C}}\} \subset (\widetilde{\Omega} G^{\mathbb{C}} \times \widetilde{\Omega} G^{\mathbb{C}})$ is the real form corresponding to the real structure

$$\tilde{\tau}(\hat{g}_1 \; , \; \hat{g}_2) = (\hat{g}_2 \; , \; \hat{g}_1) \; .$$

Imposing the additional real structure τ given by 3.80, we get the following adaptation of proposition 24 on page 88.

Proposition 41 *Let* $Lie(\widetilde{\Omega} G)^{\mathbb{C}}$ *be a complex loop algebra and denote by* $\Re Lie(\widetilde{\Omega} G^{\mathbb{C}})$ *and* $\Im Lie(\widetilde{\Omega} G^{\mathbb{C}})$ *its real and imaginary parts with respect to the real structure* τ*. Then the solutions of Nahm's system*

$$\dot{\hat{T}}_i + \frac{1}{2} \sum \varepsilon_{i,j,k} [\hat{T}_j, \hat{T}_k] = 0 \; ,$$

such that

$$\begin{array}{ll} \hat{T}_1, \hat{T}_3 & : I \longrightarrow \; \Im Lie(\widetilde{\Omega} G^{\mathbb{C}}) \\ \hat{T}_2 & : I \longrightarrow \; \Re Lie(\widetilde{\Omega} G^{\mathbb{C}}) \end{array}$$

are in one-to-one correspondence with the solutions of the variational problem on the Lie group G *given by the Lagrangian*

$$\mathcal{L}(\hat{g}) = \int_{\mathbb{R}} \|\hat{g}_t\|^2 + \widetilde{\mathcal{K}}(Ad_{\hat{g}}(\hat{\beta}) \; , \; \tilde{\hat{\beta}}) \; . \tag{3.82}$$

Taking the variation $\delta\mathcal{L}(\hat{g})$ of 3.82, a calculation similar to the one in the proof of proposition 21 shows that the Euler-Lagrange equation of 3.82 is

$$(\hat{g}_t\hat{g}^{-1})_t + [Ad_{\hat{g}}(\hat{\beta}) \ , \ \widetilde{\hat{\beta}}] \ . \tag{3.83}$$

The first summand above has the expression

$$(\hat{g}_t\hat{g}^{-1})_t = ((g_t g^{-1})_t - \dot{u}(g_s g^{-1})_t \ , \ 0)$$

where $\hat{g}(t) = (g(s) \ , \ e^u)(t)$. Applying 3.77 and denoting $\hat{\beta} = (\beta \ , \ b)$ and $\widetilde{\hat{\beta}} = (\widetilde{\beta} \ , \ \widetilde{b})$, the second summand can be written as

$$b\widetilde{b}(g_s g^{-1})_s + \mathcal{S}(\hat{\beta})$$

where

$$\mathcal{S}(\hat{\beta}) = [Ad_g(\beta(s+u)), \widetilde{\beta}] - b \cdot [g_s g^{-1}, \widetilde{\beta}] + b \cdot \widetilde{\beta}_s - \widetilde{b} \cdot \left(Ad_g(\beta(s+u))\right)_s . \tag{3.84}$$

Recall, that the Euler-Lagrange equation for the harmonic maps

$$g(t,s) : \mathbb{R}^2 \longrightarrow G$$

from the plane into a compact Lie group G is

$$(g_t g^{-1})_t + (g_s g^{-1})_s = 0 \ . \tag{3.85}$$

Interpreting the path $g(s;t) : \mathbb{R} \to \Omega G$ as the a map $g(s,t) : S^1 \times \mathbb{R} \to G$ from the cylinder into the Lie group G, the discussion above proves the following proposition.

Proposition 42 *The solutions of Nahm's system with values in the extended loop algebra and satisfying the additional condition*

$$\begin{array}{ll} \hat{T}_1, \hat{T}_3 & : I \longrightarrow \ \Im Lie(\widetilde{\Omega}G^{\mathbb{C}}) \\ \hat{T}_2 & : I \longrightarrow \ \Re Lie(\widetilde{\Omega}G^{\mathbb{C}}) \end{array}$$

are in one-to-one correspondence with the perturbations of the harmonic maps

$$g(s,t) : S^1 \times \mathbb{R} \longrightarrow G$$

given by t he Euler-Lagrange equation

$$(g_t g^{-1})_t + b\widetilde{b}(g_s g^{-1})_s + \mathcal{S}(\hat{\beta}) = 0 \ ,$$

where the perturbation $\mathcal{S}$ is given by 3.84. Taking $\hat{\beta} = \widetilde{\hat{\beta}} = (0,1)$ gives the ordinary unperturbed harmonic maps from the cylinder into the Lie group G given by the Euler-Lagrange equation 3.85.

There are many different definitions and criteria of integrability for the PDE's. One of them, which is often used in connection with the harmonic maps is the so-called zero curvature criterion.

Proposition 43 *The perturbed harmonic maps $g(s,t) : S^1 \times \mathbb{R} \to G$ given by the Euler-Lagrange equation*

$$(g_t g^{-1})_t + b\widetilde{b}(g_s g^{-1})_s + \mathcal{S}(\hat{\beta}) = 0$$

where $\mathcal{S}(\hat{\beta})$ is defined by 3.84 are integrable with respect to the zero curvature criterion.

Proof: We have already seen that Nahm's equations can be written in the form of the Lax equation. Defining $\hat{\Phi} = i\hat{T}_1 + z(\hat{T}_2 + i\hat{T}_3) - z^{-1}(\hat{T}_2 - i\hat{T}_3)$, Nahm's system is equivalent to the equation

$$\hat{\Phi}_t = [\hat{\Psi}\ ,\ \hat{\Phi}]$$

where $\hat{\Psi} = \frac{1}{2}dz(z \cdot \hat{\Phi})$. Inserting $\hat{\Phi} = \Phi + f\frac{d}{ds}$ and $\hat{\Psi} = \Psi + p\frac{d}{ds}$ into the above equation, we get

$$(\frac{1}{p} \cdot \Psi)_t + (\Phi - \frac{f}{p} \cdot \Psi)_s = [(\Phi - \frac{f}{p} \cdot \Psi)\ ,\ (\frac{1}{p} \cdot \Psi)]$$

which shows that the pair $((\Phi - \frac{f}{p} \cdot \Psi)\ ,\ (\frac{1}{p} \cdot \Psi))$ satisfies the zero curvature condition. □

Let $g : S^1 \times \mathbb{R} \to M$ be a harmonic map. The group $U(1)$ acts in a natural way on $S^1 \times \mathbb{R}$, and suppose that there is also an action of $U(1)$ on M. The Euler-Lagrange equations for an $U(1)$-equivariant map g become ODE's and hence define some dynamical system on M. In her paper [Uh] K. Uhlenbeck proves that the dynamical systems corresponding to a $U(1)$-equivariant harmonic map from the cylinder into the sphere S^n is the C. Neumann system on S^n. The question arises, what are the dynamical systems corresponding to the $U(1)$-equivariant maps $g : S^1 \times \mathbb{R} \to M$ where M is an arbitrary symmetric space. Using Nahm's equations with values in loop algebras it should not be difficult to see that the sought for systems are the Hamiltonian systems (T^*M, ω_{can}, H) discussed in this chapter.

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