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Some Results From Algebraic Graph Theory

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Nekaj rezultatov iz algebraične teorije grafov

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Izjava

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Abstract

In this thesis we present some results living in the intersection between graph theory and linear algebra. We introduce the subject of algebraic graph theory presenting some general results from this area. In particular we show how certain algebraic objects such as matrices and polynomials can be used to gain structural information about graphs. We then introduce two graph polynomials namely the chromatic polynomial and its generalization—the Tutte polynomial. We present a counterexample to a conjecture of J. Xu and Z. Liu about the chromatic polynomial and degree sequences.

We then turn our attention to matrices associated with graphs namely the adjacency matrix and distance matrix. We present some results in the context of strongly regular graphs. In particular we show a connection between graphs maximizing the number of cycles with length matching their odd girth and Moore graphs. Continuing with strongly regular graphs we present a classificational result for $(75, 32, 15, 16)$ strongly regular graphs. The approach is based on the so called star complement technique developed by Cvetković and Rowlinson.

Math. Subj. Class. (2010): `05C12`, `05C50`, `05C31`, `05C75`

Keywords: adjacency matrix, strongly regular graph, chromatic polynomial, Tutte polynomial, convex cycle

Povzetek

V disertaciji predstavimo nekaj rezultatov, ki ležijo na preseku med teorijo grafov in linearno algebro. Predstavimo področje algebraične teorije grafov in vpeljemo nekaj znanih rezultatov iz tega področja. Natančneje, pokažemo, kako nam lastnosti grafovskih polinomov in matrik določajo strukturne lastnosti ustreznih grafov. Konkretneje se osredotočimo na matriko sosednosti, razdaljno matriko in kromatični polinom. V kontekstu kromatičnega polinoma konstruiramo neskončno družino protiprimerov za domnevo J. Xu-ja in Z. Liu-ja.

V nadaljevanju disertacije se osredotočimo na pojem krepko regularnih grafov in razvijemo nekaj njihovih osnovnih lastnosti. Med drugim pokažemo tudi ekstremalno povezavo med številom konveksnih ciklov ter poddružino krepko regularnih grafov - Moorovih grafov. Konec posvetimo problemu klasifikacije krepko regularnih grafov. S pomočjo metode zvezdnega komplementa klasificiramo $(75, 32, 15, 16)$ krepko regularne grafe.

Math. Subj. Class. (2010): `05C12`, `05C50`, `05C31`, `05C75`

Ključne besede: matrika sosednosti, krepko regularen graf, kromatični polinom, Tuttov polinom, konveksni cikel

Contents

Chapter 1

Introduction

1.1 Basic graph theoretical notions

Let V be a finite set. The set of all subsets of V that have cardinality k is denoted by $\binom{V}{k}$. A *graph* G with vertex set $V(G) = V$ and edge set $E(G) = E$ is a pair (V, E) where $E \subseteq \binom{V}{2}$. While the presented definition of a graph is quite abstract, graphs are arguably among the most applicative mathematical objects, mainly due to their extensive occurrence in chemistry, social theory and most notably computer science. Indeed many practical problems can be modeled with the underlying notion of a graph. One among many is the representation of a computer network where vertices are computers and an edge represents a network connection between the two computers. Given its applicability it is not surprising that graph theory witnessed enormous development in recent years. To get an overview of some classical results in this field we refer the reader to [47], [20] and [29]. In the remaining part of this chapter we only mention the notions and results that are of relevance for our thesis.

If for two vertices $x, y \in V(G)$ the set $\{x, y\}$ belongs to $E(G)$ then we say that x and y are *adjacent* and we write $x \sim_G y$ omitting the subscript whenever the underlying graph is clear from the context. For a vertex $v \in V(G)$ we say that the *degree* of v, denoted by $d(v)$, is the number of vertices adjacent to v. The set of all such vertices is denoted by $N_G(v)$. The order and size of G are the cardinalities of $V(G)$ and $E(G)$ respectively. If u and v are vertices attaining the minimal (maximal, respectively) degree of G then we define $\delta(G) := d(u)$ and $\Delta(G) := d(v)$ respectively. If $\delta(G) = \Delta(G) = k$ then we say that G is *regular* and of valency k. An example of a regular graph is the complete graph with vertex set $V = \{1, \dots, n\}$ and $\binom{V}{2}$ as its edge set. We denote it by K_n. The fact that the underlying vertex set of K_n is $\{1, \dots, n\}$ is just for the sake of convenience and indeed any n-set would do. The following ambivalence is modeled with the notion of isomorphism. Let G and H be two graphs. If $f : V(G) \to V(H)$ is a bijection such that for any $x, y \in V(G)$ we have

$$x \sim_G y \iff f(x) \sim_H f(y)\,,$$

then we say that f is an *isomorphism* and that G and H are *isomorphic*. We write $G \cong H$. If $G = H$ then we say that f is an *automorphism* of G. The set of all automorphisms of a graph G is denoted by $\mathrm{Aut}(G)$ and forms a group under the operation of functional composition. For a graph G its *complement* is defined as the graph with the same vertex set and edge set

$$\binom{V(G)}{2} \setminus E(G)\,,$$

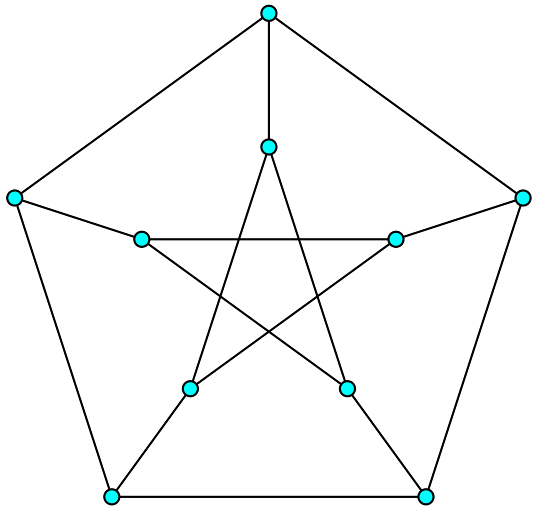


Figure 1.1: The most famous object in graph theory—the Petersen graph.

and is denoted by $\overline{G}$. If G is isomorphic to $\overline{G}$ then we say that G is *self-complementary*. If e is an edge of G then we define $G - e$ to be the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$. The graph $G - v$ where $v \in V(G)$ is defined similarly. The operation of *contraction* of an edge $e = \{x, y\}$ is denoted by G/e and is defined as the graph with vertex set $V(G - x)$ and edge set $E(G - x) \cup \{\{x, x'\} \mid x' \neq x \wedge x' \sim_G y\}$. In our thesis we will encounter another operation of constructing graphs, the so called *line graph*. The line graph of G, denoted as $L(G)$, is the graph whose vertex set is $E(G)$ and two vertices $f, f' \in V(L(G))$ are adjacent if and only if $f \cap f' \neq \emptyset$. An example of a line graph is the so called *Petersen graph* depicted on Figure 1.1. It can be seen that the Petersen graph is in fact $\overline{L(K_5)}$. Another well known construction of graphs is obtained by performing the so called *Cartesian product* operation on graphs. The Cartesian product of two graphs G and H is the graph denoted by $G \square H$ with vertex set $V(G) \times V(H)$ where adjacency is defined as

$$(u, v) \sim_{G \square H} (u', v') \iff u = u' \wedge v \sim_H v' \quad \text{or} \quad u \sim_G u' \wedge v = v' \,.$$

Example 1.1.1. A classical example of a graph obtained by the Cartesian product operation is the so called*cube graph* of dimension n which is

$$\square_{i=1}^{n} K_2 \,,$$

and is denoted by Q_n. An equivalent definition of the n-cube is to consider the graph whose vertex set is comprised of all binary strings of length n. Two vertices are adjacent if and only if the respective bit strings differ in precisely one coordinate. Figure 1.2 depicts a standard drawing of the 4-cube. Given a vertex v its *antipodal vertex* is a vertex $\overline{v}$ whose bitstring corresponds to the binary complement of the binary string represented by v.

A *subgraph* H of G is a graph with the property that $E(H) \subseteq E(G)$ and $V(H) \subseteq V(G)$. We write $H \subseteq G$. If for every pair of distinct vertices $x, y \in V(H)$ we have that $x \sim_G y \implies x \sim_H y$ then we say that H is an *induced* subgraph of G. For a set $S \subseteq V(G)$ we denote with $G[S]$ the

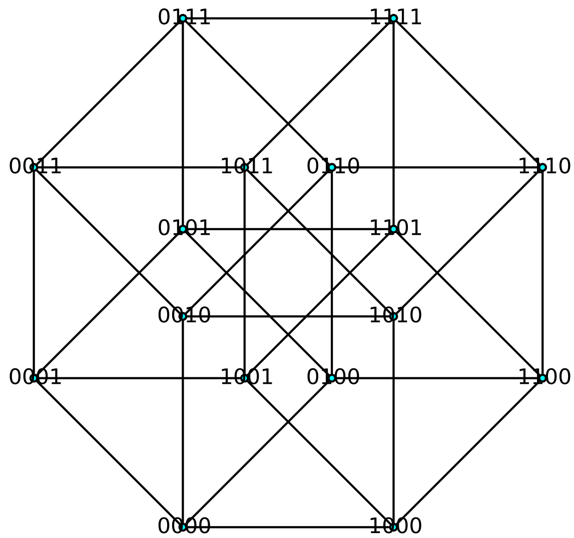


Figure 1.2: The 4-cube.

induced subgraph of G with vertex set S. If $H \subseteq G$ and H is isomorphic to a complete graph then we say that H is a *clique* of G. The largest k such that K_k is a subgraph of G is called the *clique number* of G and is denoted by $\omega(G)$. The problem of determining the clique number of a given n-vertex graph is notoriously hard even if we want to settle for an approximate solution [43]. In Chapter 4 we show how using symmetries can speed up state of the art algorithms for the clique problem.

A *walk* W in G is a sequence of vertices $v_1, \dots, v_k$ such that for every $1 \leq i < k$ we have $v_i \sim v_{i+1}$. If all the vertices of W are distinct then we call W a *path* of length $k-1$. If $v_1 = v_k$ and the vertices $v_2, \dots, v_{k-1}$ are distinct then we say that W is a *cycle* of G with length $k+1$. If for every pair of vertices u, v of G there is a path from u to v then we say that G is *connected*. The length of a shortest path from u to v is denoted by $d_G(u, v)$. The diameter of G denoted by $\mathrm{Diam}(G)$ is defined as the expression $\max_{u,v \in V(G)} d_G(u, v)$. The *girth* of G is the length of a shortest cycle in G. If G has a cycle of length n then we say that G is *hamiltonian*. If G is a connected 2-regular graph with n vertices then we call G an n-cycle and denote it as C_n. If we introduce a new vertex $c \notin V(C_n)$ and join it to every vertex of C_n we obtain the so called *wheel graph* denoted by W_{n+1}. The edges incident with the vertex of degree n are called its *spokes*.

1.1.1 Graph colorings

Suppose $c_k : V(G) \to \{1, \dots, k\}$ is a function such that for any two vertices $x, y \in V(G)$ we have $x \sim y \Rightarrow c(x) \neq c(y)$. In this case we say that c is a *proper coloring*. The least k for which G admits a proper coloring c_k is called the *chromatic number* of G and is denoted by $\chi(G)$. Since $\chi(K_n) = n$ it follows that $\chi(G) \geq \omega(G)$. While, by definition, $\chi(G) = k$ implies the existence of a proper coloring c_k for G, the coloring itself need not be unique and indeed in most cases this is not so. If one denotes by $\widehat{c}_k$ the number of proper k-colorings of G then the *chromatic polynomial* is defined as the function $P_G : \mathbb{R} \to \mathbb{R}$ with the property that for every natural number k we have

$$p_G(k) = \widehat{c}_k\,.$$

Let $x \sim y$ and let $S(G)$ be the set of all proper k-colorings of G. Notice that a proper coloring of $S(G-e)$ is a proper coloring of G if and only if x and y are assigned distinct colors. In other words $S(G) = S(G-e) \setminus S(G/e)$ and hence

Proposition 1.1.1. *If $e = \{x, y\}$ is an edge of G then $p_G(k) = p_{G-e}(k) - p_{G/e}(k)$.*

Since $p_{K_1}(k) = k$ it is not hard to see that an inductive application of Proposition 1.1.1 justifies naming p_G a polynomial.

Example 1.1.2. As mentioned, $p_{K_1}(k) = k$ and more generally $p_{K_n}(k) = k(k-1)\cdots(k-n+1)$.

Example 1.1.3. If G has precisely two connected components G_1 and G_2 then, since one can color G_1, G_2 independently, we have $p_G(k) = p_{G_1}(k)p_{G_2}(k)$.

Example 1.1.4. If T is a tree with n vertices and $e = \{u, v\}$ where u is a leaf of T then by the deletion-contraction recurrence we have $p_G(k) = k \cdot p_{T-u} - p_{T'}(k)$ where $T' = T/e$ and is a tree with $n-1$ vertices. Applying this argument on T' inductively we deduce that for any n-vertex tree T we have $p_T(k) = k(k-1)^{n-1}$. As it turns out, the converse is true as well.

Example 1.1.5. If C_n is the cycle graph and e one of its edges then $p_{C_n}(k) = p_{C_n-e} - p_{C_{n-1}}(k)$ since $C_n - e$ is a tree we can deduce from the previous example that $p_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$.

Example 1.1.6. Let v be the vertex of largest degree in a wheel W_n. If we have k available colors to color W_n and we color v with one of the k colors then there are $k-1$ remaining choices for coloring the remaining vertices of $W_n - v$. Hence $P_{W_n} = kp_{C_n}(k-1) = k(k-2)^n + (-1)^n k(k-2)$.

In Chapter 2 we establish some additional properties of $p_G(k)$.

1.2 Graph matrices and spectral graph theory

In this section we shortly review the notions and results needed in order to smoothly follow the subsequent results in this thesis. If G is a simple undirected graph with vertex set $\{v_1, \ldots, v_n\}$ then we denote by $A_G = (a_{i,j})_{i,j=1}^n$ its *adjacency matrix* which is a $n \times n$ matrix such that $a_{i,j} = 1$ if the vertices v_i and v_j are adjacent and 0 otherwise.

Let $w_{i,j}^k$ be the number of walks from v_i to v_j that have length k. The first result that we present relies on no other notion.

Proposition 1.2.1. *With the above notation we have $A^k = (w_{i,j}^k)_{i,j=1}^n$.*

Proof. Our claim admits an easy proof by induction. If $k = 0$ or $k = 1$ then the statement clearly holds. Fix $1 \le i, j \le n$. Notice that for any $k > 0$ by the definition of matrix product

$$w_{i,j}^{k+1} = \sum_{\ell=1}^{n} w_{i,\ell}^k a_{\ell,j} \,.$$

But the last expression counts the number of walks of length k to the neighbors of v_i which is precisely the number of $(k+1)$ walks from v_i to v_j. $\square$

Let us remark that Proposition 1.2.1 gives a way to compute the diameter of a graph in time complexity $\mathcal{O}(n^\omega \log^2 n)$ where $\omega < 2.373$ is the constant of the matrix multiplication algorithm. Assuming that G is connected, the diameter of G is the least natural number k such that A^k has no zero entry. In order to find the least such k we can use bisection on the set $\{1,\dots,n\}$ and in order to compute A^k we can use exponentiation by squaring which requires $\mathcal{O}(\log n)$ matrix multiplications. There are strong indications that in terms of time complexity this is in fact an optimal algorithm [66].

The matrix A_G is a real symmetric matrix implying that its eigenvalues are real. By the term *eigenvalues of* G we will mean the eigenvalues of A_G which we will denote as

$$\lambda_1(G) \geq \cdots \geq \lambda_n(G) .$$

Whenever our graph is clear from the context we will omit specifying it and simply write λ_i to denote its ith eigenvalue.

Many structural results about G can be obtained from the eigenvalues of A_G and one the first such results that we present is a result of Harary [42] dating back to the year 1962.

For a subgraph H of G let $c(H)$ be the number of connected components of H that are cycles and $r(H)$ the number of connected components that are isomorphic to K_2. Let $C_n(G)$ be the set of all spanning subgraphs of G that are the disjoint union of cycles and K_2. With this notation in mind we have:

Theorem 1.2.2. *For every simple graph G it holds*

$$\det A_G = \sum_{H \in C_n(G)} (-1)^{r(H)} 2^{c(H)} .$$

Proof. Recall that by definition

$$\det A_G = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) a_{1,\pi(1)} \cdots a_{n,\pi(n)}, \tag{1.1}$$

where S_n is the set of all permutations of $\{1,\dots,n\}$. Le us fix a permutation π such that the respective summand is nonzero. Notice that since the diagonal elements of A_G are zero it follows that π has no fixed point. Hence π can be expressed as the product of disjoint cycles whose length is at least 2. If (x,y) is a cycle of π then clearly $x \sim y$ and more generally if $(x_1, x_2, \cdots, x_k)$ is a cycle of π then for every $1 \leq i < k$ we have

$$x_i \sim x_{i+1} ,$$

as well as $x_1 \sim x_k$. In other words, the expression $\operatorname{sgn}(\pi) a_{1,\pi(1)} \cdots a_{n,\pi(n)}$ is nonzero whenever the cycles of π represent a disjoint union of K_2s and cycles of G with an implicit orientation given by π. Hence every element $H \in C_n(G)$ gives $2^{c(H)}$ summands to expression (1.1) as the direction of each of its cycles can be picked up arbitrarily. It remains to argue that $\operatorname{sgn}(\pi) = (-1)^{r(H)}$, where H is the subgraph of G corresponding to π. Let c_o, c_e be the number of connected components of H having odd and even order respectively. By definition $\operatorname{sgn}(\pi) = (-1)^{c_e}$. Now if n_i is the number of connected components of H having order i then the equation $\sum_{i=2}^n i n_i = n$ implies that $c_o \equiv n \pmod 2$. Since

$$r(H) = n - (c_e + c_o) \equiv c_e \pmod 2 ,$$

the result follows. □

While the result itself may not look useful at first glance, let us present some of its implications. Let $p(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n$ be the characteristic polynomial of A_G. Recall that the coefficient c_i is the sum of the determinant of all principal $i \times i$ submatrices of A_G. Hence we have

Proposition 1.2.3. *The coefficients of the characteristic polynomial of A_G satisfy*

$$(-1)^i c_i = \sum_{H \in C_i(G)} (-1)^{r(H)} 2^{c(H)} .$$

As observed by Sachs [68] this gives a way to determine the odd girth of G and in particular to count the respective number of such cycles.

Corollary 1.2.4. *Let G be a graph with odd girth $2r+1$ and let*

$$p(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n,$$

be the characteristic polynomial of A_G. Then

$$c_3 = c_5 = \cdots = c_{2r-1} = 0,$$

and the number of $(2r+1)$-cycles in G equals

$$-c_{2r+1}/2 .$$

In Chapter 3 we will use Corollary 1.2.4 to count the number of 5-cycles in Moore graphs. At this point let us remark that, among other things, Corollary 1.2.4 can be used to test if an n-vertex graph has a triangle. Indeed G is triangle-free if and only if $\mathrm{tr}(A^3) = 0$. Hence this gives us an algorithm for testing for triangles having time complexity asymptotic to $\mathcal{O}(n^\omega)$. As it turns out this is currently the best possible algorithm for testing triangle-free graphs. Any improvement to the matrix multiplication algorithm would thus bring an improvement for the algorithm of recognizing triangle-free graphs. Somehow surprising is the fact that the converse is true as well. Any improvement to the triangle-testing algorithm would yield an improvement for the matrix multiplication algorithm. The precise implications can be found in [74].

As expected, the adjacency matrix is not the only matrix studied in graph theory. Among others there is the so called *Laplacian matrix* defined as $L_G = \Delta_G - A_G$. Here Δ_G is the matrix whose diagonal is $d(v_1), \ldots, d(v_n)$ and all off-diagonal elements are zero. While just a slight modification of the adjacency matrix, the eigenvalues of the Laplacian matrix gives a large number of structural properties of the underlying graph. Among other things it is related to the number of spanning trees of G, maximum cuts, mean distance as well as expanding properties. For a survey on the Laplacian matrix we refer to [61]. While we will not encounter the Laplacian matrix in this thesis we will make a short excursion into the topic of the *distance matrix* of a graph. Assuming that G is a connected graph, its distance matrix $D_G = (d_{i,j})_{i,j=1}^n$ is defined as the matrix with entries $d_{i,j} = d_G(v_i, v_j)$. The structure of this matrix is much more complex than that of the adjacency matrix or the Laplacian matrix, hence there are comparably less results about its relationship with structural properties of G.

1.2.1 Interlacing

Given two sequences of real numbers $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_m$, where $n \geq m$, we say that $\{\mu_i\}_{i=1}^m$ *interlaces* $\{\lambda_i\}_{i=1}^n$ if

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \quad \text{for} \quad i \in \{1, \ldots, m\}\,.$$

The well-known interlacing principle states that the eigenvalues of an induced subgraph of G interlace the eigenvalues of G, see for example [39] or [36]. Many results in algebraic graph theory are based on the interlacing principle which sometimes gives rise to unexpected results. A well-known such example is the folklore proof that the Petersen graph is not hamiltonian. Indeed, if the Petersen graph $\mathcal{P}$ contains a cycle C of length 10 then since $L(C) = C$ the line graph of $\mathcal{P}$ must contain C as an induced subgraph. Now by computing the eigenvalues of $L(\mathcal{P})$ we see that $\lambda_7(L(\mathcal{P})) = -1$ while $\lambda_7(C) = -\frac{1}{2}\sqrt{5} + \frac{1}{2}$ and hence by the interlacing principle C is not an induced subgraph of $L(\mathcal{P})$. In other words $\mathcal{P}$ is not hamiltonian.

For our subsequent purposes we will need a more general statement that we call the partitioned interlacing criterion [39, Corollary 2.3]. We state it as follows. Suppose $\mathcal{V} = (\mathcal{V}_1, \ldots, \mathcal{V}_k)$ is a partition of the vertices of G. For $i \neq j$ let $e(\mathcal{V}_i, \mathcal{V}_j)$ denote the number of edges between the vertices of $\mathcal{V}_i$ and $\mathcal{V}_j$ if $i \neq j$, and let $e(\mathcal{V}_i)$ denote the number of edges in the graph induced by V_i. Consider the $k \times k$ matrix $A_{\mathcal{V}} = (a_{i,j})_{i,j=1}^k$, where

$$a_{i,j} = \begin{cases} \frac{e(\mathcal{V}_i,\mathcal{V}_j)}{|\mathcal{V}_i|} & \text{if} \quad i \neq j\,, \\ \frac{2e(\mathcal{V}_i)}{|\mathcal{V}_i|} & \text{if} \quad i = j\,. \end{cases}$$

As it turns out, the eigenvalues of $A_{\mathcal{V}}$ interlace the eigenvalues of G. For the sake of convenience we state the mentioned fact in the following proposition.

Proposition 1.2.5. *Let G be a graph and $\mathcal{V}$ a partition of its vertices. Then, the eigenvalues of G are interlaced by the eigenvalues of $A_{\mathcal{V}}$.*

The application of the interlacing principle is often used to determine whether a graph H is an induced subgraph of G. In cases where G and H are comparably large graphs it may be computationally infeasible to determine whether H is an induced subgraph of G. In such cases the interlacing criterion can quickly discard certain graphs H from being induced subgraphs of a target graph G. As we will see this becomes especially useful when the target graph G is in fact not known but its eigenvalues are.

Before moving to the next section let us illustrate how to use Proposition 1.2.5 in order to bound the independence number of a k-regular graph G. The first to observe the following (unpublished) property was Hoffman [35, pp. 204]

Proposition 1.2.6. *If G is a k-regular graph of order n and λ_n is its smallest eigenvalue, then*

$$\alpha(G) \leq -\frac{n\lambda_n}{k - \lambda_n}\,.$$

Proof. Let $S \subset V(G)$ be an independent set of G and consider the partition $\mathcal{V} = \{S, V(G) \setminus S\}$. Then the matrix

$$A_{\mathcal{V}} = \begin{pmatrix} 0 & k \\ \frac{k|S|}{n-|S|} & k - \frac{k|S|}{n-|S|} \end{pmatrix}$$

has eigenvalues k and $\frac{|S|k}{|S|-n}$. Proposition 1.2.5 implies that $\frac{|S|k}{|S|-n} \geq \lambda_n$ and hence the result follows. $\square$

As a final note let us remark that the above result can be generalized to non-regular graphs as well [36].

Chapter 2

The Chromatic polynomial

The notion of the chromatic polynomial dates back to 1912 and was introduced by Birkhoff [18] as a way of attacking the 4-color conjecture. The later problem is equivalent to the condition that $p_G(4) > 0$ for every planar graph G. While Birkhoff's attempt proved fruitless it nevertheless initiated a rich area of graph theory that sprouted many interesting results. From the analytic point of view he and Lewis were able to prove that for any $x \geq 5$ we have $p_G(x) > 0$ and posed the following—still open—conjecture.

Conjecture 1. *For every planar graph G and $x \in (4, 5)$ we have $p_G(x) > 0$.*

The study of the chromatic polynomial continued and in 1968 Read published his seminal paper [65] on the chromatic polynomial. Besides proving some basic properties about p_G—a subject that we explore in the next section—he also posed many interesting problems that are open to this day. Among other things he defined the notion of *chromatic equivalence*. We say that two graphs G, H are *chromatically equivalent* whenever $p_G = p_H$. A graph G is *chromatically unique* if it is determined by its chromatic polynomial. That is, if $p_G = p_H$ then G and H are isomorphic. It can be seen that an example of such a graph is the complete graph K_n and the cycle graph C_n. Another example is the class of wheel graphs W_n whenever n is odd. For n is even it is known that W_6 and W_8 are not chromatically unique (see Figure 2.1) but W_{10} is [54]. In particular it is conjectured that in fact all large enough wheels are determined by their chromatic polynomials.

Conjecture 2. *For every $n \geq 10$ the wheel graph W_n is chromatically unique.*

The reason that it is easier to establish unique chromaticity for the wheel on an odd number of vertices is the fact that such graphs are uniquely 3-colorable. That is, W_{2k+1} has only one proper coloring up to a permutation of the color classes. By the propositions established in our next section we will see that if G is chromaticaly equivalent to W_n then G must be a 2-connected graph with n vertices, $2n$ edges and n triangles. In addition the number of its induced 4-cycles must equal twice the number of its number of 4-cliques. Using McKay's Nauty [58] program with the PRUNE directive we generated all such graphs on 12 vertices. We have verified:

Proposition 2.0.7. *The wheel graphs W_{12} and W_{14} are chromatically unique.*

Removing spokes from wheels makes it easier to establish their chromatic uniqueness and there are many results in this direction, see for example [31]. The study of chromatically unique graphs is a rich field with many beautiful results. For a survey on this subject see [51] and [52].

In this chapter we first establish some known properties of the chromatic polynomial. In particular we show that the chromatic polynomial encodes certain graph invariants which are not directly

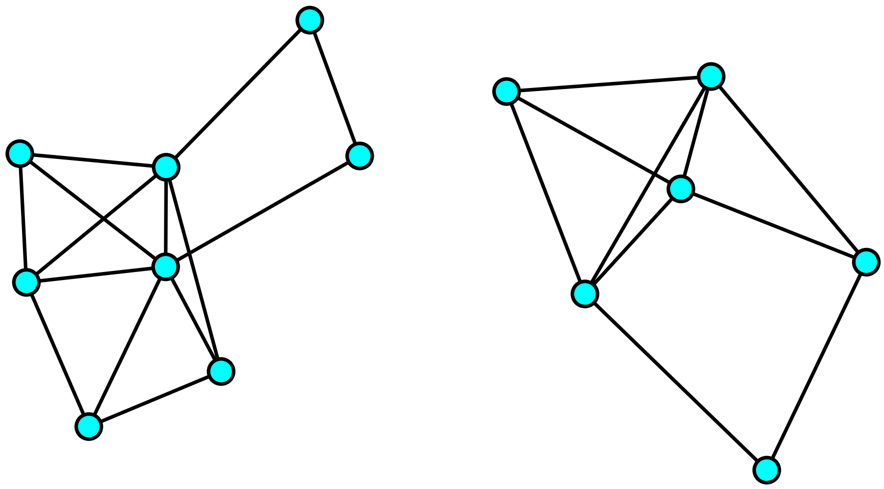


Figure 2.1: Graphs chromatically equivalent to W_8 and W_6 respectively.

related to the chromatic number. We then focus on the problem of constructing a family of graphs with the property that each graph of the family does not have the same degree sequence as its complement, yet it has the same chromatic polynomial. We finish by generalizing this result to the Tutte polynomial and posing an open problem. The described constructions were first presented in [11].

2.1 Some properties of the chromatic polynomial

In this section we briefly summarize some well-known properties of the chromatic polynomial. We specifically focus on properties related to its coefficients. Clearly, the constant term is zero and in what follows we establish what happens with the coefficients of higher degree. For ease of expression, let us denote by $c_i(G) = p_i$ where $p_G(k) = \sum_{i=1}^{n} p_i k^{n-i}$.

Proposition 2.1.1. *Let G be a graph with n vertices. Then $p_{n-1}(G) \neq 0$ if and only if G is connected.*

Proof. If G has c connected components then its chromatic polynomial is the product of the chromatic polynomials of its connected components. Since the chromatic polynomial of each connected component is divisible by k the chromatic polynomial of G must be divisible by k^c and hence the linear term coefficient of p_G is nonzero. Conversely, if G is connected and is not a tree then it has a cycle C. By taking an edge e from C we have that G/e and $G - e$ are connected graphs. Since the stated claim holds for trees it now holds for any connected graph by a direct inductive argument. $\square$

Proposition 2.1.1 can be generalized to show that if the first non-zero coefficient of $p_G(k)$ occurs at the term k^c, then G has c connected components. We continue establishing results for the coefficient of $p_G(k)$.

Proposition 2.1.2. *For every graph G we have $c_0(G) = 1$.*

Proof. The claim quickly follows by an induction argument on the number of edges of G. If $G \cong \overline{K_n}$ then the claim is obviously true. Otherwise let e be an edge of G. By the deletion-contraction recurrence the leading coefficient of $p_G(k)$ is the leading coefficient of $p_{G-e}(k)$ and hence the claim follows. □

Before establishing our next claim let us introduce an equivalent definition of the chromatic polynomial. Let $m_k(G)$ be the number of partitions of $V(G)$ into k independent sets. Then the chromatic polynomial of G satisfies

$$p_G(k) = \sum_{i=1}^{n} m_i(G) k^{\underline{i}},$$

where $k^{\underline{i}} = k(k-1)\cdots(k-i+1)$ is the falling factorial symbol. Indeed every partition of $V(G)$ into k independent sets gives rise to a proper coloring of G. Conversely for every partition into r independent sets we can color the vertices in the independent sets in $k(k-1)\cdots(k-r+1)$ ways. With this in mind we are now in the position to prove our next two claims.

Proposition 2.1.3. *If G is a n-vertex graph with m edges then the coefficient of $c_1(n) = -m$.*

Proof. By the remark above, the coefficient of k^{n-1} in $p_G(k)$ equals

$$-\binom{n}{2} m_n(G) + m_{n-1}(G).$$

Consider now a partition of $V(G)$ into $n-1$ independent sets. Clearly such a partition is comprised of $n-2$ singletons and a pair $\{x, y\}$ where $x \not\sim y$. Hence every such partition is uniquely defined by taking two non-adjacent vertices. In other words $m_{n-1} = \binom{n}{2} - m$ and hence our claim follows since $m_n(G) = 1$. □

The interpretation of the coefficient of $p_G(k)$ can be extended further.

Proposition 2.1.4. *For a graph G of size m we have $c_2(G) = \binom{m}{2} - t$, where t is the number of triangles of G.*

Proof. We prove the claim by induction on m. For $m \in \{0, 1, 2\}$ the claim clearly holds. By the deletion-contraction recurrence we have

$$c_2(G) = c_2(G-e) - c_1(G/e),$$

where e is an edge of G. Let $t_e, t_{\overline{e}}$ be the number of triangles of G containing and not containing e, respectively. By the induction hypothesis then $c_2(G-e) = \binom{m-1}{2} - t_{\overline{e}}$. The graph G/e may not be simple. In fact the contraction of e produces precisely t_e pairs of multiedges. Hence, since G/e is simple, t_e edges are removed from it. By Proposition 2.1.4, $c_1(G/e)$ encodes the number of edges of G/e and in particular $-c_1(G/e) = (m-1) - t_e$. Since $t_e + t_{\overline{e}}$ is in fact the number of triangles of G, the claim follows. □

The interpretation of the coefficients of $p_G(k)$ can be carried further although the respective expressions get more and more complicated. Let $c_4(G)$ be the number of induced 4-cycles of G and $k_4(G)$ the number of its 4-cliques. Using a slightly more involved argument, Farrell [34] has proved the following result.

Theorem 2.1.5. *The coefficient of k^{n-3} of $p_G(k)$ is*

$$-\binom{m}{3} + (m-2)t(G) + c_4(G) - 2k_4(G)\,.$$

There are also certain global results about the coefficients of the chromatic polynomial. A first result in this direction is the fact that the coefficients of $p_G(k)$ alternate in sign. The next one is easily proved by an inductive application of the deletion-contraction recurrence.

Proposition 2.1.6. *If G is a n-vertex graph and c_m is the coefficient of k^m in $p_G(k)$ then $c_m \geq 0$ if $n \equiv m \pmod 2$ and $c_m \leq 0$ otherwise.*

The study of the global properties of the coefficients of the chromatic polynomial received a lot of attention in the previous century. For example the notorious unimodal conjecture asked whether the coefficient of $p_G(k)$ in fact form a *unimodal* sequence. The problem has been posed by Read [65] in 1968 and received a lot of attention. The claim was recently established by Huh [46]. His argument is involved and uses the theory of hypersufraces.

Theorem 2.1.7. *If $p_G(k) = \sum_{i=1}^{n} a_i k^i$ is the chromatic polynomial of G then there exist a m such that*

$$|a_n| \leq |a_{n-1}| \cdots \leq |a_m| \geq |a_{m-1}| \geq \cdots \geq |a_1|\,.$$

The interpretation of the coefficients of $p_G(k)$ also offers a generalization that was first observed by Whitney [73] in 1932. Suppose the edges of G have an ordering $e_1, \ldots, e_m$. We say that $B \subseteq E(G)$ is a *broken circuit* if there is an edge $e_\ell \notin B$ such that $B \cup \{e_\ell\}$ is a cycle and $i \geq \ell$ for every i such that $e_i \in B$. With this notion in mind we have.

Theorem 2.1.8. *Suppose the edges of G are ordered and let $p_G(k) = \sum_{i=1}^{n} (-1)^i c_i(G) k^{n-i}$. Then $c_i(G)$ is the number of subgraphs of G having i edges and no broken circuits.*

We finish this section by presenting a so called *expansion* expression for the chromatic polynomial. In what follows let the function $c(F)$, where $F \subseteq E(G)$ count the number of connected components in the graph $(V(G), F)$. We have

Theorem 2.1.9. *The chromatic polynomial $p_G(k)$ satisfies the expansion formula.*

$$p_G(k) = \sum_{F \subseteq E(G)} (-1)^{|F|} k^{c(F)}\,.$$

Proof. It is enough to prove the claim for $k \in \mathbb{N}$. For an edge $e = \{u, v\}$ define the set

$$M_e = \{\kappa : V(G) \to \{1, \ldots, k\} \mid \kappa(u) = \kappa(v)\}\,.$$

By definition, the set of all proper k-colorings of G is

$$\bigcap_{e \in E(G)} \overline{M_e}\,.$$

But by the principle of inclusion-exclusion we have

$$|\bigcap_{e \in E(G)} \overline{M_e}| = \sum_{F \subseteq E(G)} (-1)^{|F|} |\bigcap_{f \in F} M_f|\,.$$

But $|\bigcap_{f \in F} M_f| = k^{c(F)}$. Indeed, a function $\kappa \in M_f$ is monochromatic on each edge of F and hence is constant on every connected component of F. Conversely for every such spanning subgraph we obtain such a function by assigning a color class to each connected component. $\square$

There are many additional results about the chromatic polynomial that are out of scope for our thesis. In fact, there is an entire book devoted to the subject of the chromatic polynomial [30] that we recommend the readers to check.

2.2 The Akiyama-Harary problem

In the late 1970's Akiyama and Harary published a series of papers [3, 4, 5, 6, 2, 8, 7, 1] initiating the study of graphs that match their complements in certain graph invariants. Their inquiry initiated a rich area of graph theory that is alive to this day [59]. Part of their exploration also involved posing open problems and in [7] they asked whether a non-self-complementary graph G can have the same chromatic polynomial as its complement. Notice that this is quite a restrictive condition since not only must G and $\overline{G}$ have the same number of edges, triangles and chromatic number but for every natural number k the number of proper k-colorings of G must be the same as the number of proper k-colorings of $\overline{G}$. Nevertheless it was confirmed by Xu and Liu [75] that such graphs indeed exists and that the smallest example has 8 vertices. They in fact constructed a family of such graphs all of them having the property that G and $\overline{G}$ share the same degree sequence. This lead them to pose the conjecture

Conjecture 3. *If a graph G has the property that $p_G(k) = p_{\overline{G}}(k)$ then G has the same degree sequence as $\overline{G}$.*

As it turns out, their conjecture is false. In what follows we present an infinite family of graphs not adhering to this condition.

Finally we turn our attention to a more general variant of this problem. For a subset $F \subseteq E(G)$ we denote by $c(F)$ the number of connected components of the graph with edge set F and vertex set $V(G)$. With this in mind the *Tutte polynomial* of a graph G is defined as

$$T_G(x, y) = \sum_{F \subseteq E(G)} (x-1)^{c(F)-c(E)} \cdot (y-1)^{c(F)+|F|-|V(G)|}. \tag{2.1}$$

The Tutte polynomial T_G contains much more information about the structure of G than p_G does. Indeed, it is a generalization of the chromatic polynomial and it is well known that

$$p_G(k) = (-1)^{|V(G)|-k(E)} k^{c(E)} T_G(1-k, 0)\,.$$

Among the many other interesting evaluations of the Tutte polynomial are $T_G(1,1)$—the number of spanning trees of G and $T_G(2,0), T_G(0,2)$ the number of cyclic and acyclic orientations of G, respectively. For a survey of known results about the Tutte polynomial see [33].

A natural generalization of the Harary-Akiyama question following from these properties of the Tutte polynomial is, whether there exists a non-self-complementary graph having the same Tutte polynomial as its complement.

2.3 Chromatic polynomials and graph complements

In this section we present a family of graphs having equal chromatic polynomials as their complements but different degree sequence. We start with the graph G_1 depicted on Figure 2.2 together with its complement. Its *graph6* string [57] is `HCpVdZY`. First, we establish that G_1 has the desired properties.

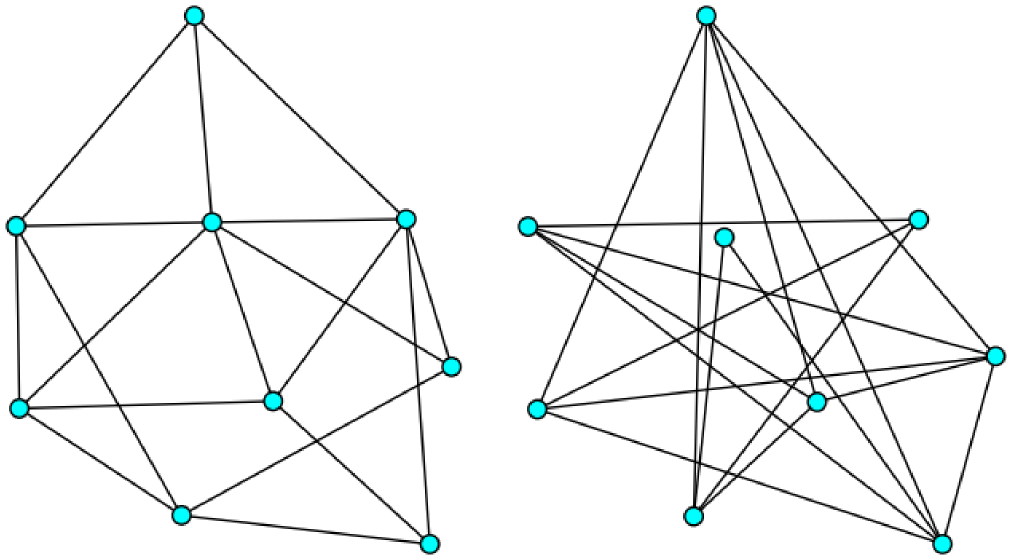


Figure 2.2: A graph and its complement.

Lemma 2.3.1. *The graph G_1 has a different degree sequence than $\overline{G_1}$ but $p_{G_1}(k) = p_{\overline{G_1}}(k)$.*

Proof. We observe that the graph G_1 from Figure 2.2 has degree sequence $(5, 5, 5, 4, 4, 4, 4, 3, 2)$ while its complement has degree sequence $(6, 5, 4, 4, 4, 4, 3, 3, 3)$. Using the well known *deletion-contraction* recurrence for computing the chromatic polynomial of a graph we can verify that:

$$p_{G_1}(k) = p_{\overline{G_1}}(k) = (k-2) \cdot (k-1) \cdot k \cdot (k-3)^2 \cdot (k^4 - 9k^3 + 35k^2 - 69k + 57)\,.$$

□

The claim of Lemma 2.3.1 can be completely verified with Sage [28] in the following way. We need to verify that the presented graph has a different degree sequence than its complement but equal chromatic polynomial. Since we have already given its graph6 string this becomes a straightforward task, as the following sequence shows.

```
sage: G = Graph('HCpVdZY')
sage: Gc = G.complement()
sage: G.degree_sequence() == Gc.degree_sequence()
False
sage: G.chromatic_polynomial() == Gc.chromatic_polynomial()
True
```

Before showing the main claim of this section, we introduce a useful construction. Given a graph G we form the graph $\widehat{G}$ by taking a vertex disjoint 4-path P and joining every vertex of G to both endpoints of P. Conveniently, we have $\overline{\widehat{G}} = \widehat{\overline{G}}$. Using this property it is not difficult to establish the following claim.

Theorem 2.3.2. *There exist infinitely many graphs G not having the same degree sequence as $\overline{G}$ but having the same chromatic polynomial as their complements.*

Proof. We compute the chromatic polynomial of $\widehat{G}$. Suppose we wish to properly color $\widehat{G}$ with k colors. Let x, y be the endpoints of the 4-path P introduced in $\widehat{G}$ and let x', y' be the respective neighbors of x and y in P. There are essentially two different ways to color $\widehat{G}$. If we color x, y with equal colors then there are $(k-1)$ choices to color x' and $(k-2)$ colors to color y' and hence $k(k-1)(k-2)p_G(k-1)$ ways to properly k-color $\widehat{G}$. If x, y are colored with different colors then we again have two cases. If y' is colored with the same color as x then we have $k(k-1)^2 p_G(k-2)$ total ways to color $\widehat{G}$. If however y' is not colored with the same color as x we end up having $k(k-1)(k-2)^2 p_G(k-2)$ ways to properly color our graph using k colors. Summing up the obtained quantities we infer

$$\begin{aligned} p_{\widehat{G}}(k) &= k(k-1)(k-2)p_G(k-1) + k(k-1)^2 p_G(k-2) + k(k-1)(k-2)^2 p_G(k-2) \\ &= k(k-1)((k-2)p_G(k-1) + (k(k-3)+3)p_G(k-2)). \end{aligned}$$

In particular we see from the above expression that $p_{\widehat{G}}(k)$ is in fact a function of $p_G(k)$. The main claim now follows quickly with an inductive argument. By Lemma 2 we have a graph G of order 9 having a different degree sequence than $\overline{G}$ but the same chromatic polynomial. But then the degree sequences of $\widehat{G}$ and $\overline{\widehat{G}}$ differ while for their chromatic polynomials the above identity implies

$$\begin{aligned} p_{\widehat{G}}(k) &= k(k-1)((k-2)p_G(k-1) + (k(k-3)+3)p_G(k-2)) \\ &= k(k-1)((k-2)p_{\overline{G}}(k-1) + (k(k-3)+3)p_{\overline{G}}(k-2)) \\ &= p_{\widehat{\overline{G}}}(k) = p_{\overline{\widehat{G}}}(k). \end{aligned}$$

Hence by using this construction iteratively we obtain an infinite family of graphs with the stated property. $\square$

By a computer search it can be seen that there are graphs on 12 vertices that have the property stated in Theorem 2.3.2. Hence it is easy to extend the proof of Theorem 2.3.2 to show that for every $n \geq 9$ congruent to 0 or 1 $\pmod 4$ there exist a graph G not having the same degree sequence as $\overline{G}$ but sharing the same chromatic polynomial.

2.4 The Tutte polynomial

A very useful property of the chromatic polynomial that we exploited in the proof of Theorem 2.3.2 is the fact that the chromatic polynomial of a graph operation is often a function of the chromatic polynomials of its operands. Unfortunately the same is not generally true for the Tutte polynomial. Indeed, consider two trees of order 4, the star graph $K_{1,3}$ and the path graph P_4. Both have the same Tutte polynomial namely x^3. Consider now their *cone graph*, that is the graph obtained by adding a new vertex and joining it to all other vertices. The cone of $K_{1,3}$ has 20 spanning trees while the cone of P_4 has 21 spanning trees. Hence the Tutte polynomials of the cones of $K_{1,3}$ and P_4 are different.

In order to apply the construction introduced in the previous section, we need an additional structure of our graphs that will assure that if two graphs G and H have equal Tutte polynomials then so do $\widehat{G}$ and $\widehat{H}$.

As it turns out, the following concept is quite useful for this purpose. Let H be a spanning subgraph of G having connected components of order $h_1 \geq h_2 \geq \cdots \geq h_k$. We say that

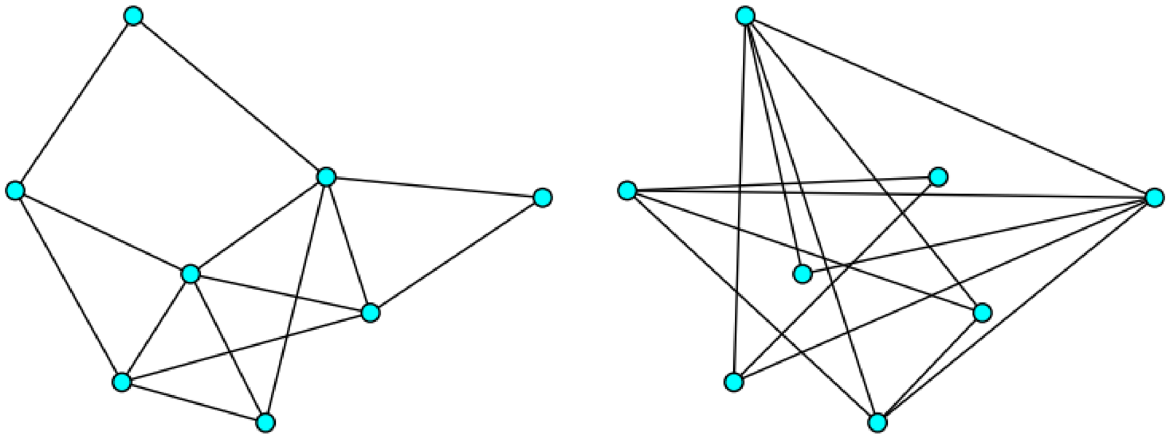


Figure 2.3: A graph whose Tutte polynomial is equal to that of its complement.

$(|E(H)|, h_1, h_2, \ldots, h_k)$ is a *subgraph description* of H. Now let $s(G)$ be the lexicographically sorted tuple of subgraph descriptions for every subgraph of G. We call $s(G)$ the *subgraph sequence* of G. Observe that equation (2.1) implies that if two graphs have the same subgraph sequence then they also have the same Tutte polynomial. The converse is of course not true as witnessed by the above example with P_4 and $K_{1,3}$. However, as our next lemma asserts, the property of having the same subgraph sequence is preserved by the construction introduced in the previous section.

Lemma 2.4.1. *If G and H are graphs such that $s(G) = s(H)$ then $s(\widehat{G}) = s(\widehat{H})$.*

Proof. Let G' be a spanning subgraph of $\widehat{G}$. Observe that G' is obtained by taking a spanning subgraph of G with subgraph description $d = (|E(G')|, g_1, \ldots, g_k)$ adding the remaining four vertices of $\widehat{G}$ coming from the introduced 4-path P and finally adding some of the edges with at least one endpoint in P. That is we add some of the edges of P and then some of the edges from the endpoints of P to some vertices of the connected components of G.

By assumption G has the same subgraph sequence as H hence there is a bijective mapping between their subgraph sequences. Let H' be the subgraph of H with subgraph sequence d that is prescribed by such bijection. Since H' and G' have the same subgraph description there is a bijective way to map every extension of G' to a subgraph of $\widehat{G}$ to an extension of H' to a subgraph of $\widehat{H}$. Indeed, we may assume the vertices of G and H to be ordered and then for every edge that is added from one of the endpoints x of P to the the ith vertex of the jth component of G we add the edge between x and the ith vertex of the jth component of H. This is always well defined since H and G have the same subgraph description. $\square$

In order to apply Lemma 2.4.1 we need to find a non self-complementary graph G such that $s(G) = s(\overline{G})$. As already noted this immediately implies $T_G(x, y) = T_{\overline{G}}(x, y)$. One of the smallest graphs with such property has order 8 and is presented on Figure 2.3. Its graph6 string is `GCRdvK` and we will denote it by H_1.

Lemma 2.4.2. *Let H_1 be the graph defined above. Then $s(H_1) = s(\overline{H_1})$ and H_1 is not self-complementary.*

Proof. Observe that H_1 and $\overline{H_1}$ both have two vertices of degree 2. In H_1 these two vertices share a common neighbor while the vertices of degree 2 in $\overline{H_1}$ have no common neighbors. Hence H_1 and $\overline{H_1}$ are not isomorphic. Verifying the second part of the claim, that is $s(H_1) = s(\overline{H_1})$, is a tedious process and we have used Sage as we describe bellow. □

In order to finalize the proof of Lemma 2.4.2 we define a function in Sage that accepts a graph G and returns the respective subgraph description.

```
def s(Gr):
    ds = []

    for A in subsets(Gr.edges()):
        G = Graph()
        G.add_vertices(Gr.vertices())
        G.add_edges(A)
        cs = [len(H) for H in G.connected_components()]
        ds.append([len(A)] + sorted(cs))

    return sorted(ds)
```

It is now a matter of a few lines to verify Lemma 2.4.1.

```
sage: G = Graph('GCRdvK')
sage: Gc = G.complement()
sage: G.is_isomorphic(Gc)
False
sage: s(G) == s(Gc)
True
```

We are now ready to prove the main claim of this section.

Theorem 2.4.3. *There exist infinitely many graphs G such that $G \ncong \overline{G}$ but $T_G(x, y) = T_{\overline{G}}(x, y)$.*

Proof. By Lemma 2.4.2 there is a non-self-complementary graph on 8 vertices such that $s(G) = s(\overline{G})$ which implies $T_G(x, y) = T_{\overline{G}}(x, y)$. But then, by Lemma 2.4.1, the graph $\widehat{G}$ again has the same subgraph description as its complement and is not self-complementary. Hence applying this operation iteratively on G we end up with an infinite family of graphs possessing the stated property. □

Again as with the chromatic polynomial we can find a graph of order 9 having the properties of Lemma 2.4.2. Hence it is possible to show in the same way as we did in the proof of Theorem 2.4.3 that for every $n \geq 8$ congruent to $0, 1 \pmod 4$ there exist a non-self-complementary graph of order n having the same Tutte polynomial as its complement.

2.5 Final remarks

We were not able to find an example of a graph G, so that G and $\overline{G}$ have different degree sequences and same Tutte polynomials. A computer search indicates that such a graph would have to have at least 16 vertices. Hence we state the following problem.

Problem 1. *Find a graph G with different degree sequence than $\overline{G}$ but same Tutte polynomial or show that such a graph does not exist.*

Interestingly the analogous problem for chromatic polynomials motivated this chapter.

Chapter 3

Strongly regular graphs

The term strongly regular graph was introduced by Bose [21] in the year 1963 in connection with the notion of partial geometries that he studied in the mentioned paper. However the notion of strong-regularity dates back at least 11 more years when the concept of association schemes was introduced [22] within the field of statistics. In graph theoretical terms an associative scheme is a distance regular graph and in particular a distance regular graph with diameter 2 corresponds to the notion of strongly regular graphs. We say that a k-regular graph of order v and diameter 2 is a *strongly regular graph*, SRG for short, with parameter set (v, k, λ, μ) if every pair of adjacent vertices has precisely λ common neighbors while two non-adjacent vertices share μ common neighbors.

In this chapter we start by presenting some well known properties of strongly regular graphs. Among other things we will compute their eigenvalues and establish certain necessary conditions for their existence. We will continue by introducing a family of strongly regular graphs named Moore graphs and show their extremality in the number of convex cycles. The later was part of the work published in [12]. We finish the chapter by showing that the distance matrix of certain strongly regular graphs has more positive than negative eigenvalues. This was part of a question posed in [37] and the answer appeared in [10].

Before moving on to our next section, let us give a series of examples that should get us acquainted with the notion of strongly regular graphs.

Example 3.0.1. The smallest example of a strongly regular graph is the 5-cycle having parameter set $(5, 2, 0, 1)$. The next natural example is of course the Petersen graph having parameter set $(10, 0, 3, 1)$. Both examples fall into the category of Moore graphs which we present in more detail in Section 3.2.

Example 3.0.2. Consider the graph obtained by taking the 4-cube Q_4 and adding edges between pairs of antipodal vertices. Figure 3.1 depicts the obtained graph. It is easy to see that the obtained graph is in fact a $(16, 5, 0, 2)$ strongly regular graph. This is the unique $(16, 5, 0, 2)$ SRG and is known under the name *Clebsch graph*.

Example 3.0.3. If X is a SRG with parameter set (v, k, λ, μ) then $\overline{X}$ is again a SRG whose parameters are $(v, v-k-1, v-2-2k+\mu, v-2k+\lambda)$.

Example 3.0.4. For an infinite family of strongly regular graphs one can take the so called *rook graphs* which are obtained by taking the Cartesian product $K_n \square K_n$. Clearly the graph has n^2 vertices and is $2n-2$ regular. Two adjacent vertices have $n-2$ common neighbors (to be found in the same fiber). If x, y are non-adjacent vertices then clearly have exactly 2 common neighbors. Hence $K_n \square K_n$ is a SRG with parameter set $(n^2, 2n-2, n-2, 2)$.

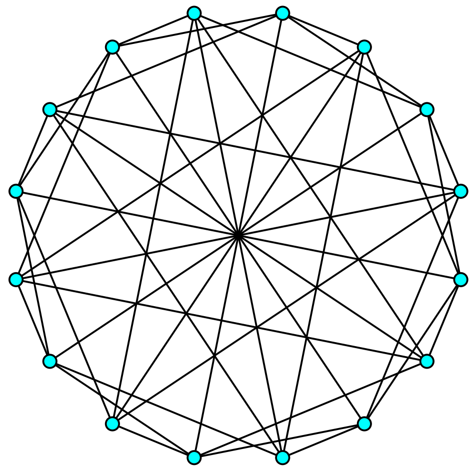


Figure 3.1: The Clebsch graph.

Example 3.0.5. The triangular graph T_n is defined as the line graph of K_n. For example T_3 is the triangle, T_4 is the so called *octahedral graph* while T_5 is the complement of the Petersen graph. It is not hard to see that T_n has $n(n-1)/2$ vertices and is $2(n-2)$ regular. If x, y, z are distinct elements of $\{1, \ldots, n\}$ then the vertices $\{x, y\}, \{x, z\}$ have $\{x, z\}$ as a common vertex as well as every vertex of the form $\{x, i\}$ where $i \notin \{y, z\}$. Hence by symmetry every two adjacent vertices of T_n have $n-2$ common neighbors. Finally two non-adjacent vertices $\{x, y\}$ and $\{z, w\}$ have exactly four common neighbors, namely $\{x, z\}, \{x, w\}, \{y, z\}$ and $\{y, w\}$. Hence T_n is a strongly regular graph with parameter set $(n(n-1)/2, 2(n-2), n-2, 4)$.

Example 3.0.6. Let q be a prime power such that $q \equiv 1 \pmod 4$. The *Paley graph* $\mathrm{P}(q)$ is defined as the graph whose vertices are the elements of $\mathrm{GF}(q)$, Two distinct vertices are adjacent whenever the difference of two vertices is a square. It can be verified that Paley graphs are in fact strongly regular with parameter set $(q, (q-1)/2, (q-5)/4, (q-1)/4)$. If q is a prime then the corresponding Paley graphs fall into the category of so called *circulant graphs*. Paley graphs have many interesting properties. Among other things, they are self-complementary and are connected with the notion of expander graphs and quasi-random graphs [25]. In addition, the study of their clique numbers is a notorious number theoretical problem [45].

A SRG with the parameter set $(v, (v-1)/2, (v-5)/4, (v-1)/4)$ is called a *conference graph* due to its connection with the notion of *conference matrices* [16]. As we can see from Example 3.0.6 every Paley graph is in fact a conference graph. However, the converse is not true. The smallest number of vertices for which the existence of a conference graph is not known is 65 and we record this fact in the following problem.

Problem 2. *Is there a conference graph with* 65 *vertices?*

Notice that Problem 2 in fact asks for the existence of a SRG with parameter set $(65, 32, 15, 16)$ which coincidentally is also the smallest parameter set whose feasibility—at the time of writing—is not known.

As indicated by the above problem, a fundamental question about strongly regular graphs is for which parameter sets does a strongly regular graph exist? For example the notorious question about the existence of a graph of order 3250, girth 5, and diameter 2 (also known as Moore graph) asks for the existence of a strongly regular graph with parameter set $(3250, 57, 0, 1)$. We say that a parameter set (v, k, λ, μ) is *feasible* whenever there exists a (v, k, λ, μ) strongly regular graph. A parameter set that is not feasible will be called *infeasible*. In our next section we present some necessary conditions on v, k, λ and μ that rule out some parameter sets.

These conditions still leave room for many parameters for which it is not known whether there exists such a SRG. The state of affairs for all possible parameters on up to 1300 vertices is tracked by Brouwer on his web site [23]. It can be seen that on up to 100 vertices there are essentially 15 parameter sets whose classification is still open, the smallest three being $(65, 32, 15, 16)$, $(69, 20, 7, 5)$, and $(75, 32, 10, 16)$. Given that there is no general technique for deciding whether a certain parameter is feasible, a lot of effort has been put into establishing certain structural results about the missing SRGs. Specifically for a potential SRG X with parameters $(75, 32, 10, 16)$, Haemers and Tonchev [40] showed in 1996 that the chromatic number of X is at least 6. Four years later Makhnev showed [56] that X does not contain a 16-regular subgraph. Recently Behbahani and Lam [15] also derived some constraints about the structure of the automorphism group of X. Particularly, they showed that if p is a prime dividing $|\operatorname{Aut}(X)|$, then $p = 2$ or $p = 3$. In Chapter 4 we show that the parameter set $(75, 32, 10, 16)$ is in fact infeasible.

3.1 Some basic properties of strongly regular graphs

In this section we present some well known properties of strongly regular graphs. As a first example we present an elementary feasibility condition. If X is a SRG and $x \in V(X)$ then we denote by $N_1(x)$ and $N_2(x)$ the vertices at distance 1 and 2 respectively from x. The sets $N_1(x), N_2(x)$ are also called the *first and second subconstitutents* of X with respect to x.

Proposition 3.1.1. *If X is a (v, k, λ, μ) SRG then*

$$(v - k - 1)\mu = k(k - \lambda - 1)\,.$$

Proof. Let $x \in V(X)$. We count the number of edges from $N_1(x)$ to $N_2(x)$ in two ways. A vertex $y \in N_1(x)$ is adjacent to x and hence must have precisely λ neighbors in $N_1(x)$ and therefore $k - \lambda - 1$ neighbors in $N_2(x)$. Since $|N_1(x)| = k$ we deduce that there are $k(k - \lambda - 1)$ edges from $N_1(x)$ to $N_2(x)$. On the other hand every vertex $y \in N_2(x)$ is not adjacent to x and hence sends μ edges to $N_2(x)$. Since the diameter of X is 2 we have $|N_2(x)| = |V(X)| - k - 1 = v - k - 1$ and thus the result follows by the double counting principle. □

Proposition 3.1.1 gives us our first feasibility condition.

Example 3.1.1. The parameter set $(30, 14, 3, 7)$ is not feasible. Indeed

$$(30 - 14 - 1) \cdot 7 \neq 14 \cdot (14 - 3 - 1)\,,$$

which is in contradiction with Proposition3.1.1.

Our second feasibility criterion comes from computing the eigenvalues of a SRG. In order to prove the claim we will need the following lemma.

Lemma 3.1.2. *Let A be a real symmetric matrix and u, v two of its eigenvectors that correspond to distinct eigenvalues of A. Then $u^T v = 0$ that is, u and v are orthogonal.*

Proof. Let $Au = \lambda u$ and $Av = \lambda v$. Since A is symmetric we have

$$u^T Av = (v^T Au)^T .$$

But then $\mu u^T v = \lambda u^T v$ and since $\mu \neq \lambda$ it must follow that $u^T v = 0$ as claimed. □

Proposition 3.1.3. *The eigenvalues of a SRG with parameter set (v, k, λ, μ) are*

$$k,\ \frac{1}{2}\left[(\lambda-\mu)+\sqrt{(\lambda-\mu)^2+4(k-\mu)}\right] \quad \textit{and} \quad \frac{1}{2}\left[(\lambda-\mu)-\sqrt{(\lambda-\mu)^2+4(k-\mu)}\right] ,$$

with respective multiplicities

$$1,\ \frac{1}{2}\left[(v-1)-\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right] \quad \textit{and} \quad \frac{1}{2}\left[(v-1)+\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right] .$$

Proof. It is not hard to see that every connected k-regular graph has k as an eigenvalue with multiplicity 1 and the respective eigenvector is the all-ones vector $\vec{1}$, see [17, p. 14]. If X is a (v, k, λ, μ) SRG then and $x, y \in V(X)$ are two distinct vertices then the number of 2-walks from x to y is μ if $x \not\sim y$ and λ if $x \sim y$. Hence by Proposition 1.2.1 we have the following identity

$$A^2 + (\mu-\lambda)A + (\lambda-k)I = \mu J. \tag{3.1}$$

If u is an eigenvector for $r \neq k$ then by multiplying Equation (3.1) with u and applying Lemma 3.1.2 we get that $r^2 + (\mu-\lambda)r + (\mu-k) = 0$ and hence (after solving the quadratic equation) that the other possible eigenvalues of A are

$$\frac{1}{2}((\lambda-\mu) \pm \sqrt{(\lambda-\mu)^2+4(k-\mu)}) .$$

Let us now denote the eigenvalues of A_X by

$$r = \frac{1}{2}((\lambda-\mu) + \sqrt{(\lambda-\mu)^2+4(k-\mu)}) ,$$

and

$$s = \frac{1}{2}((\lambda-\mu) - \sqrt{(\lambda-\mu)^2+4(k-\mu)}) ,$$

with the respective multiplicities denoted by f, g. We note the following two relations

$$f + g = v - 1 \quad \text{and} \quad rf + sg + k = 0 ,$$

the first equation following from the fact that k has multiplicity 1 and the second from the fact that the sum of the eigenvalues is the trace of A_X. We have obtained two linear equations in two unknowns and their solution is

$$f = \frac{1}{2}\left(v-1-\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right)$$

and

$$g = \frac{1}{2}\left(v-1+\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right) ,$$

as claimed. □

This already gives some additional infeasibility conditions since the multiplicities must be integers as we see in our next example.

Example 3.1.2. Consider the parameter set $(40, 18, 10, 6)$. Clearly $(40-18-1)\cdot 6 = 18\cdot(18-10-1)$ but $\sqrt{(10-6)^2+4\cdot(18-6)}$ is irrational and hence the eigenvalues of such a SRG could not have integral multiplicities.

There are now two ways to extend Proposition 3.1.3. One is to show that in terms of eigenvalues, strongly regular graphs are precisely the regular graphs having 3 distinct eigenvalues. In terms of integrality we can also tell when their spectrum is integral. We give a precise formulation of both these claims in the next two propositions.

Proposition 3.1.4. *A connected regular graph is strongly regular if and only if it has precisely three distinct eigenvalues.*

Proof. Let G be a connected k-regular graph that has precisely three eigenvalues k, r and s. For every vector orthogonal to $\vec{1}$ we have that

$$(A - rI)(A - sI)\cdot v = 0\,,$$

and therefore $(A-rI)(A-sI) = \alpha J$ for some $\alpha \in \mathbb{R}$. But then

$$A^2 = (r+s+\alpha)A + \alpha(J - A - I) + (rs+\alpha)I,$$

and hence the number of common neighbors between any two vertices of G only depends on whether they are adjacent or not. Hence G is strongly regular. $\square$

As mentioned we can also describe the situation in which the eigenvalues of a SRG are integers. As we shall see this happens whenever the multiplicities of the non-trivial eigenvalues differ. Before stating the claim we need a short lemma.

Lemma 3.1.5. *If x is a rational number such that*

$$x^2 + bx + c = 0\,,$$

for integers b, c then x is an integer.

Proof. Assume $x = \frac{p}{q}$ where p and q are coprime. We have

$$(p/q)^2 + b(p/q) + c = 0$$

and therefore $p^2 + bpq + cq^2 = 0$. It now follows that y divides p^2 and therefore $q = 1$ since p and q are coprime. $\square$

Proposition 3.1.6. *If f, g are the multiplicities of the non-trivial eigenvalues of a SRG with parameter set (v, k, λ, μ) then its eigenvalues are integers, unless $f = g$.*

Proof. Let the corresponding eigenvalues be r, s. From Proposition 3.1.3 we deduce

$$r + s \quad = \quad \lambda + \mu. \tag{3.2}$$

Finally since the trace of the adjacency matrix of a graph is 0 we have

$$k + fr + gs = 0. \tag{3.3}$$

If $f \neq g$ then Equations (3.2) and (3.3) are linearly independent and hence determine r and s uniquely. In particular since the two equations have integer coefficients it implies that both r and s are rational. Finally since r and s were obtained as zeros of a quadratic equation it follows from Lemma 3.1.5 that they are in fact integers. □

In addition to the conditions that we just presented there are two other necessary conditions for the feasibility of a parameter set. They are referred under the name *Krein bound* and *absolute bound*. The former is usually stated in the more general theory of association schemes. Before stating the two conditions let us implicitly assume that our SRG has eigenvalues $k > r > s$ with respective multiplicities $1, f$ and g. With this in mind we can state Krein condition as follows.

Theorem 3.1.7. (Krein Condition) *For a strongly regular graph G with parameter set* (v, k, λ, μ) *we have*

$$(r + 1)(k + r + 2rs) \leq (k + r)(s + 1)^2$$

and

$$(s + 1)(k + s + 2rs) \leq (k + s)(r + 1)^2 \,.$$

Example 3.1.3. Consider the parameter set $(184, 48, 2, 16)$. Clearly

$$(184 - 48 - 1) \cdot 16 = 2160 = 48 \cdot (48 - 2 - 1)\,.$$

Similarly, the necessary conditions for the integrality of the eigenvalues show that such a graph would have eigenvalues $48, 2, -16$ with respective multiplicities $1, 160$ and 23. It is not hard to verify that the second inequality of the Krein condition is not satisfied and hence that the parameter set $(184, 48, 2, 16)$ is in fact not feasible.

The absolute bound is related to bounds on the spherical two-distance sets in $\mathbb{R}^f$ and $\mathbb{R}^g$. Its formulation in the context of strongly regular graphs is as follows.

Theorem 3.1.8. (Absolute Bound) *For a SRG with parameter set* (v, k, λ, μ) *G we have*

$$v \leq \frac{1}{2} f(f + 3) \quad \textit{and} \quad v \leq \frac{1}{2} g(g + 3)\,.$$

Example 3.1.4. It can be verified that the parameter set $(50, 21, 4, 12)$ satisfies all the presented necessary conditions. However since -9 would have to be an eigenvalue with multiplicity 7 it would imply $50 \leq \frac{1}{2}7(7 + 3) = 35$ and thus contradicting the absolute bound.

3.2 Moore graphs and convexity in graph theory

We say that a *Moore graph* is a graph of diameter d and girth $2d + 1$. Their name was attributed by Hoffman and Singleton in honor of E.F. Moore who first raised the problem of classifying such graphs. Soon it was established [69] that every such graph must in fact be regular and hence has precisely

$$M(d, k) = 1 + k \sum_{i=0}^{d-1} (k - 1)^i,$$

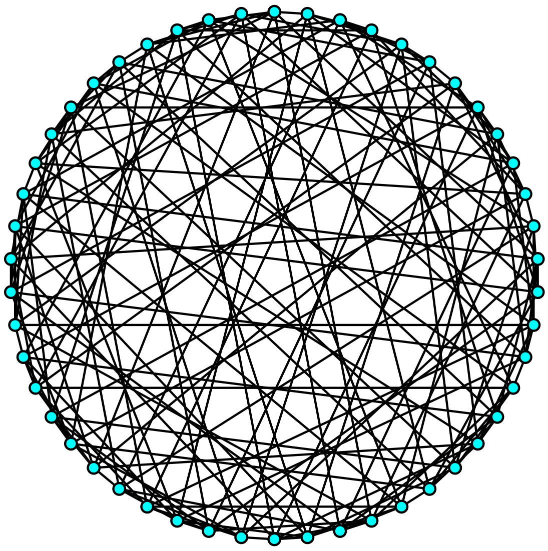


Figure 3.2: A Moore graph on 50 vertices - the Hoffman-Singleton graph.

vertices where k is its valency. In fact another way to classify Moore graphs is to define them as k-regular graphs with diameter d having precisely $M(d,k)$ vertices.

If $d \geq 3$ then Bannai, Ito [14] and (independently) Damerell [27] established that there is no Moore graph with the exception of the cycle of length $2d+1$. Hence, every other Moore graph must have diameter 2 and thus share its membership with the Petersen graph and the 5-cycle. Hoffman and Singleton found an additional Moore graph on 50 vertices depicted on Figure 3.2, and also established that the list of possible valencies for such a graph is finite. Part of their result is easily established by the theory developed in Section 3.1.

Proposition 3.2.1. *If G is a Moore graph with diameter 2 and valency k, then $k \in \{2,3,7,57\}$.*

Proof. Notice that G is necessarily strongly regular. Indeed if x, y are two adjacent vertices then since the graph is triangle-free they share no common vertex. If $x \not\sim y$ then since the graph has diameter 2 we must have $|N(x) \cap N(y)| \geq 1$ but the fact that such Moore graph are C_4-free implies that $|N(x) \cap N(y)| = 1$. Hence every k-regular Moore graph is in fact a SRG with parameter set $(1+k^2, k, 0, 1)$. From Lemma 3.1.3 we have that the non-trivial eigenvalues of G have multiplicities

$$\frac{1}{2}(k^2 \pm \frac{2k-k^2}{\sqrt{1+4(k-1)}})\,. \tag{3.4}$$

Clearly $k=2$ implies integrality of (3.4) and in particular, this is the only possibility where the expression $\sqrt{1+4(k-1)}$ is allowed to be irrational. Otherwise $k=(p^2+3)/4$ for some integer p. Simplifying (3.4) with this in mind we obtain

$$\frac{(15+2p^2-p^4)}{16p}\,. \tag{3.5}$$

must be an integer. Now integrality of (3.5) implies that $p|15$ and hence $p \in \{1, 3, 5, 15\}$ so that in this case $k \in \{1, 3, 7, 57\}$. Since $k = 1$ forces G to be the complete graph it follows that $k \in \{2, 3, 7, 57\}$ and the proof is complete. □

It is not hard to see that for $k = 2, 3$ the only Moore graphs are the 5-cycle and the Petersen graph. For $k = 7$ Hoffman and Singleton provided a construction of the unique such graph with 50 vertices and valency 7.

Notice that from the proof of Proposition 3.2.1 every Moore graph of valency k is a strongly regular graph with parameter set $(1 + (k-1) + (k-1)^2, k, 0, 1)$. From the above summary we can infer that the only uncovered case is the existence of a $(3250, 57, 0, 1)$ SRG. The existence of such a graph is unknown and is, according to [35], the most famous open problem in algebraic graph theory [35].

Problem 3. *Is there a* $(3250, 57, 0, 1)$ *strongly regular graph?*

While it is not even known whether such a graph would in fact be unique, there are some structural results about its automorphism group. As it can easily be seen the 5-cycle, the Petersen graph as well as the Hoffman-Singleton graph are all vertex transitive. Somehow surprisingly it was first proved by Higman that a SRG with parameter set $(3250, 57, 0, 1)$, if it eexists, is not vertex transitive. In fact, Mačaj and Šíran proved [55] that its automorphism group has order at most 375.

Before presenting the connection between Moore graphs and convex cycles let us mention another problem related to Problem 3. The 5-cycle, Petersen graph, Clebsch graph and the Hoffman Singleton graph are all triangle-free and so is a potential Moore graph on 3250 vertices. In addition, there are three more known triangle-free strongly regular graphs namely the Gewritz graph $(56, 10, 0, 2)$, the Higman-Sims graph $(100, 22, 0, 6)$ and a SRG with parameter set $(77, 16, 0, 4)$. Despite much effort it is not known whether there exist any additional triangle-free SRG. Hence we record the following problem.

Problem 4. *Is there an eighth triangle-free SRG?*

The smallest open parameter for a triangle-free SRG is $(162, 21, 0, 3)$.

In our next section we show yet another characterization of Moore graphs in terms of their number of cycles of odd girth. We will show that Moore graphs and odd cycles are the only graphs with girth $g = 2d + 1$ having $n \cdot (m - n + 1)/g$ cycles of length g where n, m are their respective order and size. We present the result in a more general setting involving the notion of *convex cycles*.

3.3 Convex cycles

Let H be a subgraph of G. We say that H is *convex* (as a subgraph of G) if for any $u, v \in V(H)$, every shortest u, v-path in G lies completely in H. In particular, if H is convex in G, then $d_H(u, v) = d_G(u, v)$ holds for every $u, v \in V(H)$. Convex subgraphs and in particular convex cycles are often employed to unveil additional structure of the studied graphs especially in the field of (Cartesian) graph products. Extending a result of Vanden Cruyce [72] for hypercubes, Egawa [32] characterized Cartesian products of complete graphs by convex subgraphs while Chepoi [24] characterized isometric subgraphs of Cartesian products of complete graphs via convexity of certain subgraphs. In [13] weak Cartesian products of trees are characterized among median graphs by the property that $K_{2,3}$ without an edge is not a convex subgraph. As a final instance of convexity in relation to product graphs let us mention that the classical unique prime factorization theorem with respect to the Cartesian product, admits a short proof if one makes use of convexity. See [41] for details.

Among convex subgraphs, convex cycles are frequently studied. In [63] Polat proved that a netlike partial cube is prism-retractable if and only if it contains at most one convex cycle of length greater than 4 while in [64] he showed that any netlike partial cube that is without an isometric ray contains a convex cycle or a finite hypercube which is fixed by every automorphism. Parallel to the above mentioned Polat's result it was proved in [49] that a partial cube is almost-median if and only if it contains no convex cycle of length greater than 4. Very recently the convex excess of a graph was introduced as the sum of contributions of all of its convex cycles and used to obtain an inequality involving the order, the size, the isometric dimension, and the convex excess of an arbitrary partial cube [50].

In the next section we consider convex cycles from an extremal point of view: what is the largest number of convex cycles that a given graph can have? We became interested in this question because of the recent paper [44] by Hellmuth, Leydold, and Stadler in which convex cycle bases are studied. Along the way they also proved that a graph G of order n and size m contains at most nm convex cycles. In the next section we strengthen their result in an extremal sense.

The remaining part of this section is organized as follows. First we bound the number of odd convex cycles of a given graph and prove that precisely the Moore graphs are extremal graphs. Following this we establish a corresponding upper bound for even convex cycles and finally derive a combined inequality.

In what follows G will denote a simple graph on n vertices, m edges, and of girth $g \geq 3$. The following characterization of convex cycles is a modification of a related result proved in [44]. More precisely, the first part (for odd cycles) is the same, while the second part is modified to serve our purposes.

Lemma 3.3.1. *Let C be a cycle of G. If $|C| = 2k+1$, $k \geq 1$, then C is convex if and only if for every edge $e = xy$ of C there exists a vertex $v \in C$ such that*

(i) $d_G(x,v) = d_G(y,v) = k$, *and*

(ii) *the x,v-path (resp. y,v-path) on C of length k is a unique shortest x,v-path (resp. y,v-path) in G.*

If $|C| = 2k$, $k \geq 2$, then C is convex if and only if for every vertex $u \in C$ there exists a vertex $v \in C$ such that

(iii) $d_G(u,v) = k$,

(iv) *there are precisely two u,v-paths in G of length k.*

Proof. As mentioned above, we only need to prove the even case. Hence let $|C| = 2k$, $k \geq 2$. It is clear that the two conditions are necessary. Suppose now that for every vertex $u \in C$ there exists a vertex $v \in C$ such that (iii) and (iv) hold. By way of contradiction assume that there are vertices $x, y \in C$ such that there is a shortest x,y-path P that is not completely contained in C. Let x' be the vertex on C with $d_G(x,x') = k$. By (iv) there are precisely two x,x'-paths in G of length k and they are both contained in C. Then y belongs to one of these paths, denote it with Q. If P is shorter than the length of the x,y-subpath of Q, then $d_G(x,x') < k$, a contradiction. And if P is of the same length as the x,y-subpath of Q, then we would have at least three x,x'-paths of length k, which contradicts (iv) for x and x'. □

Note that if follows from the first part of Lemma 3.3.1 that in a graph of girth $g = 2r + 1$ all of its g-cycles are convex. We will call a pair $(e, v) \in E(G) \times V(G)$ that satisfies conditions (i) and (ii) of Lemma 3.3.1 an *odd antipodal pair*. Likewise if $(u, v) \in V(G) \times V(G)$ satisfies conditions (iii) and (iv) then we will say that (u, v) is an *even antipodal pair*. In cases where the context is clear we will simply say that a pair (a, b) is *antipodal* if it is an even or odd antipodal pair. Observe that Lemma 3.3.1 readily implies that the number of odd convex cycles is $\mathcal{O}(nm)$ while the number of even convex cycles is $\mathcal{O}(n^2)$. In what follows we give sharper estimates for these two quantities by bounding the number of antipodal pairs.

Lemma 3.3.2. *Let $v \in V(G)$ there exist at most $m - n + 1$ edges e such that (e, v) is an odd antipodal pair.*

Proof. Let T be a BFS tree of G with root v and let $e \in E(T)$, as one endpoint of e is closer to v than the other, (e, v) is not antipodal. □

From Lemma 3.3.2 we get an estimate on the number of odd convex cycles in G, which we denote by $\rho_o(G)$.

Lemma 3.3.3. *Let G be a graph with n vertices, m edges, and girth g. Then the number of its odd convex cycles $\rho_o(G)$ satisfies*

$$\rho_o(G) \leq \frac{n}{g}(m - n + 1)\,. \tag{3.6}$$

Proof. Suppose that G contains k odd convex cycles. Every convex cycle C determines precisely $|C| \geq g$ antipodal pairs. We select one and assign it to C. Doing it for every convex cycle, there are at least $k(g-1)$ antipodal pairs that are not assigned to convex cycles. In addition, by Lemma 3.3.2, a vertex of G does not form an antipodal pair with at least $n - 1$ edges. Therefore we have at least $n(n - 1)$ non-antipodal pairs. If follows that

$$k \leq nm - k(g - 1) - n(n - 1)\,,$$

and thus

$$k \leq \frac{n}{g}(m - n + 1)\,,$$

as claimed. □

If G is a cycle, then $m = n = g$, thus the bound of Lemma 3.3.3 is sharp for all odd cycles. The same holds for complete graphs K_n, $n \geq 3$. Indeed, for K_n we have $g = 3$, $m = \binom{n}{2}$, and any triple of vertices induces a triangle, hence the assertion follows because $\frac{n}{3}\left(\binom{n}{2} - n + 1\right) = \binom{n}{3}$. We next show that equality (3.6) holds precisely for Moore graphs. Before stating the claim let us recall that by Proposition 3.1.3 the characteristic polynomial of the Hoffman-Singleton graph is $(x - 7)(x + 3)^{21}(x - 2)^{28}$ while a Moore graph with parameter set $(3250, 57, 0, 1)$, provided it exists, has characteristic polynomial equal to $(x - 57)(x + 8)^{1520}(x - 7)^{1729}$.

Lemma 3.3.4. $\rho_o(G) = \frac{n}{g}(m - n + 1)$ *if and only if G is a Moore graph.*

Proof. Suppose first that G is a graph that satisfies the equality. Then it follows from Lemma 3.3.3 and its proof that the girth g of G is odd and that all convex cycles of G are of length $g = 2r + 1$. Recall from Lemma 3.3.2 that a vertex $v \in V(G)$ lies in at most $m - n + 1$ antipodal pairs. Since the equality is satisfied for G, it follows that every edge which is not on a BFS tree with a root v

constitutes an antipodal pair with v. In other words every such edge joins two vertices x, y such that $d_G(v,x) = d_G(v,y) = r$. Consider now a BFS tree T rooted at v and let v' be a leaf of T. Observe that v' has degree at least two in G because $\rho_o(G) = \rho_o(G-u)$ holds for any pendant vertex u. Hence there is an edge e not in T that is adjacent to v' in G. From the above remark it follows that (e, v) is an antipodal pair and therefore $d_G(v, v') = r$. This in turn implies that G has diameter r. Since the girth of G is $2r+1$ we conclude that G is a Moore graph.

To prove the converse we need to show that every Moore graph satisfies equality (3.6). As already observed, this is the case with odd cycles and complete graphs of order $n \geq 3$. The Petersen graph has girth 5, hence all of its twelve 5-cycles are convex. Since $\frac{10}{5}(15-10+1) = 12$, the bound for the Petersen graph is established. It thus remains to show that the Hoffman-Singleton graph H and a possible Moore graph X of diameter 2 and degree 57 also have the claimed property. To show this we simply invoke Corollary 1.2.4. The Hoffman-Singleton graph H has 50 vertices, 175 edges, and $p_H(x) = (x-7)(x-2)^{28}(x+3)^{21}$. Since it has girth 5 and

$$\frac{\left(\frac{d^{45}}{dx^{45}}p_H(x)\right)(0)}{45!} = -2520\,,$$

it follows that the number of 5-cycles of H is 1260. Hence the bound of Lemma 3.3.4 is sharp for H. Similarly for a possible Moore graph X we have that $p_X(x) = (x-57)(x+8)^{1520}(x-7)^{1729}$. Since the coefficient of x^{3245} in the polynomial $p_X(x)$ is -116188800 it follows that X has 58094400 5-cycles. Given the fact that X has degree 57 and order 3250, it is now straightforward to verify that X also satisfies the equality. $\square$

3.3.1 Even convex cycles

We next derive an upper bound for the number of even convex cycles, denoted with $\rho_e(G)$. The bound is similar to the above bound for $\rho_o(G)$.

It follows from the second part of Lemma 3.3.1 that if (v, v') is an even antipodal pair then $d_G(v, v') \geq 2$. Combining this with the fact that every even convex cycle C yields $|C|/2$ antipodal pairs, gives the bound

$$\rho_e(G) \leq \frac{n(n-1)-2m}{g}\,.$$

While this bound is of the right order, it is not very sharp for sparse graphs. The next result establishes a better bound for graphs with a small cyclomatic number, that is, with a small $m-n+1$.

Lemma 3.3.5. *Let G be a graph with n vertices, m edges and girth g. The number of its even convex cycles $\rho_e(G)$ satisfies*

$$\rho_e(G) \leq \frac{n}{g}(m-n+1)\,. \tag{3.7}$$

Moreover, equality holds if and only if G is an even cycle.

Proof. We claim that every vertex $v \in V(G)$ lies in at most $m-n+1$ even antipodal pairs. Let (v, v') be an antipodal pair of vertices from an even convex cycle C. Let T be a BFS tree rooted at v. Lemma 3.3.1 implies that all the edges of C are on T with the exception of one edge e that is incident with v' on C. So for every vertex v' that is antipodal with v there is at least one edge e not on T that is adjacent to v'. This proves the claim. In total we therefore have at most $n(m-n+1)$

even antipodal pairs. In addition, every even convex cycle of length $2k$ yields k antipodal pairs. Since we only need to count unordered pairs, we deduce that

$$\rho_e(G) \leq \frac{n}{g}(m - n + 1)\,.$$

For the equality part of the lemma, let C be an even convex cycle of G. If $G = C$ then equality clearly holds. Otherwise, let u be a vertex of G that is not on C and is adjacent to a vertex $v \in C$. Let v' be the antipodal vertex of v on C. Then observe that (u, v') is not an antipodal pair. Moreover, at least one edge that is incident with v' on C is not on a BFS tree rooted at u. We deduce that u is contained in less than $m - n + 1$ even antipodal pairs which implies that G has less than $\frac{n}{g}(m - n + 1)$ even convex cycles. □

3.3.2 A combined inequality

We finally combine the derived bounds for $\rho_o(G)$ and $\rho_e(G)$ into a single inequality for the number $\rho(G)$ of all convex cycles of G. The key insight is that graphs with the maximum number of convex cycles are homogeneous in the sense that they either contain only even or only odd convex cycles. The following lemma establishes this fact.

Lemma 3.3.6. $\rho(G) \leq \frac{n}{g}(m - n + 1)$. *Moreover, if G contains an even convex cycle then the bound is attained if and only if $G = C_n$.*

Proof. Suppose that C is an even convex cycle of G. Let $v \in C$ and consider a BFS tree T rooted at v. Let v' be the antipodal vertex of v with respect to C. Let e and f be the edges of C incident with v'. Then none of e, f forms an antipodal pair with v as the end-vertices of e (or f) are not at the same distance to v. This means that for every even convex cycle there is at least one less possible odd convex cycle which in turn implies that

$$\rho(G) \leq \frac{n}{g}(m - n + 1)\,.$$

Suppose now that G contains an even convex cycle C and that $G \neq C_n$. Let $u \notin C$ be a vertex of G that is adjacent to a vertex $v \in C$. Let v' be the antipodal vertex of v on C and consider a shortest u, v'-path $P_{uv'}$. We distinguish two cases and wish to show that the given configuration forbids the attainment of the bound.

Case 1. $P_{uv'} \cap C \neq \emptyset$.

In this case (u, v') is not an antipodal pair of an even convex cycle. Moreover at least one edge incident with v' on C is not in a BFS tree rooted at u and also does not form an antipodal pair with u. The latter fact implies that $\rho(G) < \frac{n}{g}(m - n + 1)$.

Case 2. $P_{uv'} \cap C = \emptyset$.

In this case the degree of v' is at least 3 and, because C is convex, $|P_{uv'}| = d_G(v, v')$. It follows that a BFS tree T rooted at u does not contain the edges e and f that are on C incident with v. Moreover, none of these two edges forms an antipodal pair with u. Since (u, v') is an antipodal pair of at most one even convex cycle, u is contained in strictly less than $m - n + 1$ antipodal pairs. Therefore the inequality for $\rho(G)$ is again not attained. □

By combining Lemma 3.3.6 with the results of the previous subsections we are now in the position to state the main theorem of this section.

Theorem 3.3.7. *Let G be a simple graph of order n, size m, and girth $g \geq 3$. Then G contains at most*

$$\frac{n}{g}(m-n+1)\,,$$

convex cycles. Moreover, equality holds if and only if G is an even cycle or a Moore graph.

3.4 Distance matrices and short embeddings

In 1971 Graham and Polak [38] studied a problem related to a switching task performed at Bell Systems in the context of telephone networks. The underlying problem was modeled as follows. Consider two strings $x = x_1 \cdots x_k$ and $y = y_1 \cdots y_k$ from $\{0, 1, \star\}^k$ and define

$$h(x, y) = \mid \{(x_i, y_i) | 1 \leq i \leq k \wedge \{x_i, y_i\} = \{0, 1\}\} \mid \;.$$

That is $h(x, y)$ counts the number of positions such that the corresponding coordinates of x and y are 0 and 1. Given a graph G we say that $\ell_k : V(G) \to \{0, 1, \star\}^k$ is a *distance labeling* if for any $u, v \in V(G)$ we have

$$d(u, v) = h(\ell_k(u), \ell_k(v))\,.$$

Observe that for large enough k such a labeling always exists and we denote by $N(G)$ the least number k such that there exist a distance labeling ℓ_k of G.

Example 3.4.1. If Q_n is the n-cube then $N(Q_n) = n$.

Example 3.4.2. For a tree T we can construct a distance labeling that does not use the star symbol as follows. Inductively, let ℓ' be a distance labeling of $T' = T - v$ where v is a leaf of T. Let $\ell(u) = 0\ell'(u)$ for all $u \neq v$ and $\ell(v) = 1\ell'(v)$. Clearly, ℓ is a distance labeling of T and hence $N(T) \leq |V(T)| - 1$ since for the tree on 2 vertices we can obtain a distance labeling with length 1.

Given the examples above it may be compelling to conjecture that $N(G) \leq |V(G)| - 1$. This is indeed the case and the proof of this fact earned by Peter Winker $100$$ as promised by Graham, who conjectured this statement.

While Graham and Polak were not able to prove this inequality they still established certain results related to $N(G)$, most notably an upper bound related with the number of negative and positive eigenvalues of D_G. More precisely, the following claim was established.

Theorem 3.4.1. *Let G be a graph and $n_-(G), n_+(G)$ the number of negative and positive eigenvalues of D_G. Then*

$$N(G) \;\geq\; \max\{n_-(G), n_+(G)\}\,. \tag{3.8}$$

In the same paper they have established that the determinant of the distance matrix of a tree depends only on its number of vertices. Since then many new proofs of this fact arose and at this point we present a particularly elegant proof due to Yan and Yeh [76]. The proof relies on the so called Dodgson's determinant evaluation rule. For a $n \times n$ matrix M and subsets $C, R \subset \{1, \ldots, n\}$

let M_R^C be the matrix obtained from M by removing the columns indexed by C and rows indexed by R. Dogdson determinant rule then states that the following equality holds

$$\det(M)\det(M_{1,n}^{1,n}) = \det(M_1^1)\det(M_n^n) - \det(M_1^n)\det(M_n^1), \tag{3.9}$$

where we removed set notations in order to simplify the expression. One can find a half-page combinatorial proof of (3.9) due to Zeilberger [77]. Given this identity we are able to prove our next claim.

Theorem 3.4.2. *If T is a tree on n vertices then*

$$\det(D(T)) = -(n-1)(-2)^{n-2}.$$

Proof. If $n = 1, 2, 3$ then the claim clearly holds. Let us therefore assume that the claim is true for every tree of order $k < n$ and let T be a tree of order n. Let D be the distance matrix of T and let two leaves of T be indexed by the columns v_1, v_n in D. By the induction hypothesis we have

$$\det(D_{1,n}^{1,n}) = -(n-3)(-2)^{n-4} \quad \text{and} \quad \det(D_1^1) = \det(D_n^n) = -(n-2)(-2)^{n-3}.$$

Hence by using (3.9) we have

$$\det(D)\cdot(-(n-3)(-2)^{n-4}) = (-(n-2)(-2)^{n-3})^2 - \det(D_1^n)^2. \tag{3.10}$$

In order to compute D_1^n we need an additional equation. We note that if v_1 is a column of the leaf and its neighbor respectively, then $v_1 - v_1' = (-1, 1, \ldots, 1)$. Hence we have that $(v_1 - v_1' + v_n' - v_n) = (-2, 0, \ldots, 0, 2)$ and thus using the Laplace expansion on the matrix obtained by performing the described elementary column operations we infer

$$\det(D) = 2(n-2)(-2)^{n-3} + 2(-1)^{n+1}\det(D_1^n). \tag{3.11}$$

Now by solving the quadratic equation produced by (3.10) and (3.11) we infer that

$$\det(D_1^n) = 2^{n-2}$$

and hence that

$$\det(D) = -(n-1)(-2)^{n-2}$$

as claimed. □

Inspecting small graphs one can see that $n_-(G) > n_+(G)$ which, if true, would further simplify bound (3.8). In fact in a subsequent paper [37] Graham and Lovász remarked that it is not known if in fact a graph satisfying $n_-(G) \leq n_+(G)$ actually exist. In our next section we show their existence. Given their nature we will call them *optimistic graphs*.

3.4.1 Eigenvalues of the distance matrix of strongly regular graphs

It can be verified by a straightforward Sage program that there is no optimistic graph on up to 11 vertices. However, we show that there is such a graph on 13 vertices. In particular, we have

Theorem 3.4.3. *Every conference graph of order $v > 9$ is optimistic.*

Proof. Let G be a strongly regular graph with parameter set (v, k, λ, μ) and distance matrix D. Let u, v be two distinct vertices of G. By definition, the number of 2-walks between u and v is λ if u and v are adjacent and μ otherwise. If A denotes the *adjacency* matrix of G then, since G has diameter 2, its distance matrix is

$$D = A + \frac{2}{\mu} \cdot (A^2 - kI - \lambda A) = \frac{2}{\mu} A^2 + (1 - \frac{2\lambda}{\mu})A - \frac{2k}{\mu} I. \tag{3.12}$$

For a conference graph this simplifies further to

$$D = \frac{1}{v-1}(8A^2 + (9-v)A) - 4I\,.$$

Since this is a polynomial in A we obtain the eigenvalues of D by plugging the eigenvalues of a SRG in the above equation. We thus infer that if G is a conference graph its eigenvalues are

$$\frac{3}{2}(v-1),\ \frac{-3-\sqrt{v}}{2},\ \frac{-3+\sqrt{v}}{2},$$

and since for $v > 9$ precisely $\frac{v+1}{2}$ of the eigenvalues are positive we deduce our claim. $\square$

Before commenting the result we state the relation derived in the above proof that can be used to compute the eigenvalues of the distance matrix of a strongly regular graph.

Proposition 3.4.4. *Let G be a strongly regular graph with parameter set (v, k, λ, μ), Then ν is an eigenvalue of A_G if and only if $\frac{2}{\mu}\nu^2 + (1 - \frac{2\lambda}{\mu})\nu - \frac{2k}{\mu}$ is an eigenvalue of D_G.*

As mentioned in the introductory examples, Paley graphs are representatives of conference graphs. We can find additional optimistic graphs by extending our search from here in virtually every direction. For example there are many other optimistic strongly regular graphs, one of them being the Hall-Janko graph $\mathcal{H}$ with parameters $(100, 36, 14, 12)$. It can be verified using (3.12) that $n_+(\mathcal{H}) = n_-(\mathcal{H}) + 28$.

There are additional self-complementary optimistic graphs as well. A non-regular example is shown on Figure 3.3. It can be checked that (excluding Paley graphs) there are precisely 6 additional optimistic self-complementary graphs on up to 17 vertices, all of them satisfying the relation $n_+(G) = n_-(G) + 1$ and having diameter 2. However there are examples of optimistic graphs of higher diameter as well. The smallest vertex-transitive optimistic graphs having diameter 3 and 4 respectively, are depicted on Figure 3.4. Their graph6 representation being

```
UsaCC@u]QwLODoIo@wBI?So?{??@~??lw?h{?Bv?
```

and

```
YsP@?__C?A?O@@AA?GOCA?C??_G?g?@O?G??@?????o_?Cc???S???g_.
```

In all our examples the gap between the number of positive and negative eigenvalues of the distance matrix is rather small. For example the gap for Paley graphs is 1. However, it is not hard to find graphs such that the gap between $n_+(G)$ and $n_-(G)$ is arbitrarily large. During the author's work on this problem one of the referees asked whether one can construct graphs of order n such that

$$n_+(G) - n_-(G) \geq c \log n$$

for some constant $c > 0$. As it turns out one can obtain an even larger gap of order $\Theta(n)$.

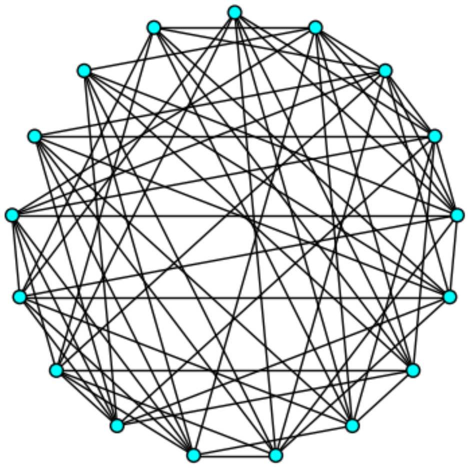


Figure 3.3: An optimistic self-complementary graph with *graph6* string representation `P?BMP{}kmh[X\\SjCrHisfYJ[`

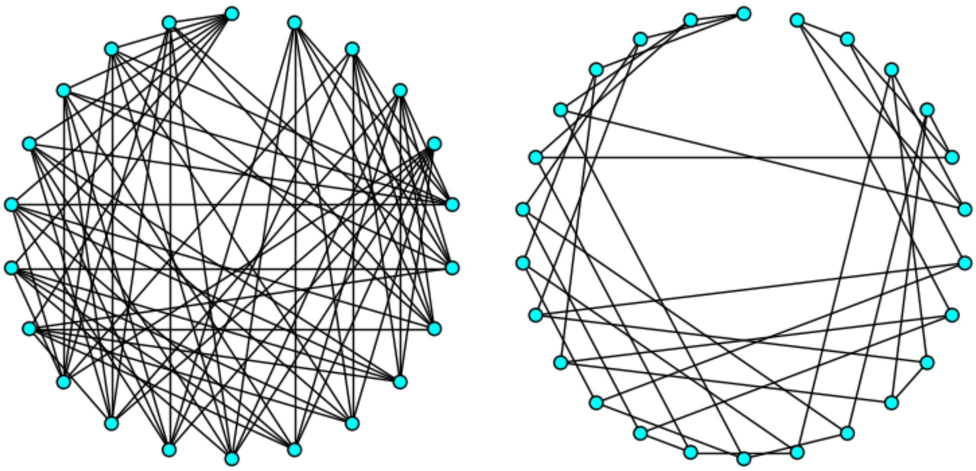


Figure 3.4: Smallest vertex-transitive optimistic graphs of diameter 3 and 4 respectively.

It is well known that there exists a strongly regular graph with parameters $(m^2, 3(m-1), m, 6)$ for every $m > 2$. From Proposition 3.4.4 we can deduce that the eigenvalues of its distance matrix are $1, 1-m, m(2m-3)+1$ with respective multiplicities $m^2-3m+2, 3m-3, 1$. Hence for every such graph G we have

$$n_+(G) - n_-(G) = m^2 - 6m + 6\,.$$

As remarked at the beginning of the section a computer search indicated that there is no optimistic graph on at most 11 vertices. Since there are too many graphs of order 12 to be inspected we leave the following for further research.

Problem 5. *Is there an optimistic graph of order* 12*? If not, is the Paley graph of order* 13 *the unique optimistic graph on* 13 *vertices?*

Our construction of an infinite sequence of optimistic graphs relied on the fact that it is very easy to compute the distance matrix of a graph of diameter 2. While we exhibited some concrete examples of higher diameter, it would be interesting to see a family of optimistic graphs with increasing diameter.

Problem 6. *Construct a family of optimistic graphs with arbitrarily large diameter.*

Chapter 4

Infeasibility of the parameter set $(75, 32, 10, 16)$

In this chapter we show how to rule out the existence of a strongly regular graph with parameter set $(75, 32, 10, 16)$. This was of particular importance since the graph is linked to certain construction of 2-graphs as well as partial ordered geometries [36].

In order to establish this result we will need to make use of a the so called star-complement technique introduced by Cvetković and Rowlinson [26]. The star-complement technique turned out to be a useful tool for re-proving classification results for some SRGs [60, 70] although its direct application fails for SRGs with a large number of vertices or large valency. The two main drawbacks being the large search space for induced subgraphs and the problem of computing the clique number of some large graphs. In this chapter we will present methods to overcome both drawbacks.

This chapter is organized as follows. In our next section we first give an overview of the star complement technique. Specifically we present the part of the theory that is suitable for our result. We then show a structural result about a SRG with parameter set $(75, 32, 10, 16)$. More precisely we show that its clique number must be 5 and that every 4-clique is contained in a 5-clique. Establishing this claim is a long and technical result. The main idea is to construct small induced subgraphs of a SRG with parameter set $(75, 32, 10, 16)$ and using the theory of star complements. We continue by showing that the parameter set $(75, 32, 10, 16)$ is not feasible by applying the fact that its clique number is 5 and using the implications that this claim brings along. We conclude the chapter by discussing some of the computational aspects of our approach. In addition we show how to take symmetries into account when solving the maximal clique problem.

4.1 Star complements

The idea behind star-complements revolves around the notion of a so called *star-complement graph*. Let G be a simple graph of order n, A_G its adjacency matrix and r one of its eigenvalues with multiplicity f. We will say that r is an eigenvalue of G whenever we mean that r is an eigenvalue of A_G.

An induced subgraph $H \subseteq G$ is called a *star-complement for G and eigenvalue* r if it has order $n - f$ and r is not an eigenvalue of H. As it turns out [26], there is a star-complement for every eigenvalue of G. For convenience we record this fact in the following proposition.

Proposition 4.1.1. *If G is a graph and r an eigenvalue of G, then G has a star complement for r.*

Before explaining the role of star-complements, let us mention that one can construct a star-complement for an eigenvalue r by extending an induced subgraph of G that does not contain r as an eigenvalue [60, Lemma 3]. More precisely:

Proposition 4.1.2. *Let G be a graph with eigenvalue r. If H' is an induced subgraph of G that does not contain r as an eigenvalue then there exist a star complement H for G and eigenvalue r so that H' is an induced subgraph of H.*

The main motivation of star-complements is that they in some way allow us to reconstruct G. The reader can find the precise implications in [26, pp. 150], in this chapter we shall formulate the theory to suit the needs of our application.

Let H be a star-complement for G with eigenvalue r and define the product

$$\langle u, v\rangle = u(rI - A_H)^{-1}v^t\,.$$

The *comparability graph* of H and r denoted by $\mathrm{Comp}(H, r)$ is the graph with vertex set

$$V(\mathrm{Comp}(H, r)) = \{u \in \{0, 1\}^{n-f} \mid \langle u, u\rangle = r \text{ and } \langle u, \overrightarrow{1}\rangle = -1\}\,,$$

and adjacency defined as

$$u \sim v \iff \langle u, v\rangle \in \{-1, 0\}\,.$$

Let us remark that the condition that the inner product of a vertex of $\mathrm{Comp}(H, r)$ and the all-ones vector is -1 does not hold in general but only if we assume that G is a regular graph, see [67].

As it turns out, the problem of constructing G is reduced to the problem of finding cliques in $\mathrm{Comp}(H, r)$. Specifically

Proposition 4.1.3. *If r is an eigenvalue of G with multiplicity f, H a star complement for G and r, then $\mathrm{Comp}(H, r)$ has a f-clique.*

This already sets the general idea behind the application of the star complement technique. Suppose G is a SRG with parameter set (v, k, λ, μ) and r an eigenvalue of G with (large) multiplicity f. Suppose that we know that H' is a induced subgraph of G and does not have r as an eigenvalue. If H' is large enough we can compute its comparability graph and check for f-cliques. If the obtained graph does not have such a clique then G does not exist.

In most cases we cannot directly find an induced subgraph H' large enough to be a star complement. In that case, by Proposition 4.1.2, we can extend H' in all possible ways to obtain candidates for a star-complement of G and r. Depending on how large H' is, we may obtain a large set of candidates, and for each such candidate H we need to compute the respective clique number of $\mathrm{Comp}(H, r)$.

The set of all possible candidates for star-complements gets large very quickly and hence we need an efficient pruning method. As it turns out the partitioned interlacing principle introduced in Section 1.2.1 provides this condition.

4.1.1 Proof outline

The problem of determining whether a (v, k, λ, μ) SRG graph G exists is thus reduced to the following. Pick an eigenvalue r of G with large multiplicity. Start with a large induced subgraph H' that does not have r as an eigenvalue and must appear as an induced subgraph of G. Extend H' to a star-complement of G and r using the described pruning conditions to get rid of invalid graphs. Finally, for all potential star-complements H compute the clique number of $\mathrm{Comp}(H, r)$. In practice, $\mathrm{Comp}(H, r)$ can be a very large and dense graph and we explain how to compute its clique number in Section 4.4.

Now, let us describe our approach for the classification of SRG with parameter set $(75, 32, 10, 16)$. For the eigenvalue we take $r = 2$ and look to find a small list $\mathcal{L}$ with graphs of large order such that at least one member of $\mathcal{L}$ is an induced subgraph of a $(75, 32, 10, 16)$ SRG X_{75}. When the list is obtained, we proceed to show that no graph in $\mathcal{L}$ is an induced subgraph of X_{75} as follows. For $H \in \mathcal{L}$ let $\mathrm{sc}(H)$ be a largest induced subgraph of H that does not have 2 as an eigenvalue and has order at most 19. Note that $\mathrm{sc}(H)$ may not be unique and in this case we can pick an arbitrary such subgraph. If $|V(\mathrm{sc}(H))| = 19$ then $\mathrm{sc}(H)$ is a star complement for X_{75} and we use the theory described above to verify that $\omega(\mathrm{Comp}(\mathrm{sc}(H), 2)) < 56$, and hence that H is not an induced subgraph of X_{75}. If $|V(\mathrm{sc}(H))| < 19$ then we extend $\mathrm{sc}(H)$ by adding $19 - |V(\mathrm{sc}(H))|$ vertices in all possible ways as to obtain (by Proposition 4.1.2) a list of possible star complements for X_{75}. Again, we show that none of the obtained star complements has a comparability graph with a large enough clique. The process of extending an induced subgraph H to a graph of order 19 is done by inductively introducing new vertices in all possible ways, and in the end removing all candidates that have 2 as eigenvalue or do not interlace. In order to minimize the list of candidate graphs we also make use of the following observation. Suppose that there is a pair of vertices $u, v \in V(H)$ that does not yet have many common neighbors in the induced subgraph - that is $u \sim v$ and $|N(u) \cap N(v)| < \lambda = 10$, or $u \not\sim v$ and $|N(u) \cap N(v)| < \mu = 16$. Suppose further that for every $S \subset V(H) \setminus \{u, v\}$ all the graphs obtained by adding a new vertex adjacent to $S \cup \{u, v\}$ that interlace X_{75} also do not contain 2 as an eigenvalue. Let us say that such a pair of vertices is called *graceful*.

In virtue of Proposition 4.1.2 we can simply use these graphs when building a complete list of star complements of X_{75} having H as subgraph. Stating it as a proposition

Proposition 4.1.4. *If u, v is a graceful pair for H and $\mathcal{L}$ a list of all graphs obtained by adding a new vertex x to H that is joined to u, v and a subset of $V(H) \setminus \{u, v\}$. Then there exist a star complement G for X_{75} such that at least one of the members of $\mathcal{L}$ is an induced subgraph of G.*

The described approach is performed by a tailor made C program *extend.c* that we describe later. In particular, it turns out that the above procedure is computationally feasible if the list $\mathcal{L}$ of induced subgraphs does not include graphs that are, when reduced to a subgraph without eigenvalue 2, of order less than 17. In practice, this is almost the same as demanding that for each $G \in \mathcal{L}$ we have $n(G) - k_2(G) \geq 17$, where $n(G)$ is order of G and $k_2(G)$ is the multiplicity of eigenvalue 2 in G.

4.2 Cliques in a SRG with parameter set $(75, 32, 10, 16)$

In what follows let X_{75} denote a possible strongly regular graph with parameters $(75, 32, 10, 16)$. Our main goal is to prove that X_{75} does not exits. In order to do so we first establish a structural claim related to its cliques. Notice that Hoffman bound [35, pp. 204] implies that $\overline{X_{75}}$ has independence number at most 5 and hence that X_{75} has clique number at most 5. On the other hand, Bondarenko, Prymak, and Radchenko developed a general tool for bounding the number of 4-cliques in a strongly regular graph [19]. In particular, they have established that a SRG with parameters $(75, 32, 10, 16)$ has at least 783 4-cliques.

In this section we show that in fact X_{75} has clique number 5, more precisely, we show the following result.

Proposition 4.2.1. *If X_{75} exists, its clique number is 5. Moreover, every 4-clique of X_{75} is contained in a 5-clique.*

In order to prove the result we need to recall a very useful lemma whose proof the reader may find in [19]. Let H be an induced subgraph of order m of a (v, k, λ, μ) strongly regular graph G, and let $(d_0, d_1, \dots, d_{m-1})$ be a vector such that d_i denotes the number of vertices of H having degree i. Similarly let $(b_0, \dots, b_m)$ be a vector where b_i denotes the number of vertices of $G - H$ that have i neighbors in H. The next lemma gives a relationship between these numbers.

Lemma 4.2.2. *With notation as above, the following three equations hold*

$$\begin{aligned} \sum_{i=0}^{m} b_i &= v - m\,, \\ \sum_{i=0}^{m} i b_i &= mk - \sum_{i=0}^{m-1} j d_j\,, \\ \sum_{i=0}^{m} \binom{i}{2} b_i &= \binom{m}{2}\mu - \sum_{i=0}^{m-1} \binom{i}{2} d_i + \frac{1}{2}(\lambda - \mu) \sum_{i=0}^{m-1} i d_i\,. \end{aligned} \tag{4.1}$$

Suppose now that X_{75} has a 4-clique H that is not contained in a 5-clique. Applying the above Lemma it can easily be verified that there are 4 solutions (b_0, b_1, b_2, b_3) (notice that by our assumption $b_4 = 0$) to the above system, namely

$$(3, 20, 48, 0), (0, 29, 39, 3), (1, 26, 42, 2) \text{ and } (2, 23, 45, 1). \tag{4.2}$$

In what follows we analyze all these possibilities, showing that none of these solutions occurs as a configuration in X_{75}. We split the proof into four sections each dealing with a different value of (b_0, b_1, b_2, b_3). The general idea is to use the structure given by a specific configuration to find a small list of graphs that must be an induced subgraph of X_{75}. For each possible case we have written simple Sage [28] programs that build graphs with the established structure and prune them using the interlacing principle we described. Whenever we assert that some induced structure is not possible or say that there is a list of graphs satisfying it, there is a corresponding Sage program that computed this part of the claim. Each such program (with the respective output) is recorded in Table 4.1. Throughout the rest of the chapter H will denote a 4-clique of X_{75}.

In the next section we shall analyze the case when X_{75} has clique number 5. Since we will also need a glimpse into this case already in this section, let us remark at this point that if K_5 is a 5-clique of X_{75} then every vertex in $V(X_{75}) - V(K_5)$ has precisely two neighbors in K_5. One way to see this claim is to use the above system of equities which only gives $(0, 0, 70, 0, 0, 0)$ as a solution vector. Finally, let us remark that throughout the chapter we will use the notation $X_{75}[S]$ to denote the subgraph of X_{75} induced by the set of vertices $S \subseteq V(X_{75})$.

4.2.1 **Case** $(3, 20, 48, 0)$

Let us denote with X_0, X_1, X_2 the subsets of vertices in $V(X_{75}) \setminus V(H)$ that have, respectively, 0,1, and 2 neighbors in H. Moreover, denote the vertices in X_0 by x_1, x_2, x_3.

Lemma 4.2.3. *Every vertex in X_2 has precisely two neighbors in X_0.*

Proof. Let $x_i \in X_0$. We use an argument that will be repeatedly used in this chapter. Since x_i is not adjacent to any of the vertices in H it has to have 16 common neighbors (since X_{75} is strongly regular with $\mu = 16$) with each of its vertices. Thus there are $4 \cdot 16$ paths of length 2 from x_i to H. On the other hand, x_i has 32 neighbors (X_{75} is 32-regular) in $X_0 \cup X_1 \cup X_2$. All the neighbors are in fact in X_2, for otherwise they could not form 64 2-paths to H. Since $|N(x_i) \cap N(x_j)| = 16$ and $|N(x_i)| = 32$, we have

$$48 \geq |N(x_1) \cup N(x_2) \cup N(x_3)| = 3 \cdot 32 - 3 \cdot 16 + |N(x_1) \cap N(x_2) \cap N(x_3)|,$$

by the inclusion-exclusion principle. Thus $N(x_1) \cap N(x_2) \cap N(x_3) = \emptyset$. Therefore every vertex in X_2 is adjacent to precisely two vertices in X_0. □

For $i \in \{1, 2, 3\}$ let $X_2^{i,j} \subseteq X_2$ be the graphs induced by $N(x_i) \cap N(x_j)$.

Lemma 4.2.4. *The graphs $X_{1,2}, X_{1,3}, X_{2,3}$ have order 16 and are isomorphic to the disjoint union of cycles.*

Proof. It is clear that the sets have order 16. Let $v \in X_{1,2}$. We count the number of 2-paths from v to H. Since v is adjacent to 2 vertices of H, there must be $2 \cdot 10 + 2 \cdot 16$ such paths. Denote with k the number of neighbors of v in X_2. By the previous lemma, v is adjacent to 2 vertices in X_0, thus it is adjacent to $32 - 2 - 2 - k$ vertices in X_1. By counting 2-paths we thus have:

$$2 \cdot 10 + 2 \cdot 16 = 2k + 1(28 - k) + 0 \cdot 2 + 2 \cdot 3\,.$$

Therefore, v has 18 neighbors in X_2, and since it is not adjacent to x_3 it must have 16 neighbors in $X_2 - X^{1,2} = N(x_3)$. This implies that v has precisely 2 neighbors in $X_{1,2}$. □

In [56] it was shown that if X_{75} exists, then it does not have a 16-regular subgraph. By removing the disjoint union of cycles from the graph $X_{75}[X_2]$ we obtain a 16-regular subgraph and hence this configuration is impossible.

4.2.2 **Case** $(1, 26, 42, 2)$

Let X_0, X_1, X_2, X_3 be the sets of vertices having $0, 1, 2$, and 3 neighbors in H, respectively. In particular, let $x_0 \in X_0$ and $x_1 \neq x_2 \in X_3$. In what follows we prove a series of claim describing the structure of a graph with this configuration.

Lemma 4.2.5. *Vertices x_1 and x_2 are not adjacent.*

Proof. Suppose $x_1 \sim x_2$. There are up to isomorphism only two possible induced graphs on $H \cup \{x_1, x_2\}$. Moreover, if we add the vertex x_0 we obtain 6 candidate graphs for an induced subgraph of X_{75}. None of them interlaces X_{75}, which was an easy task to check by computer. □

Lemma 4.2.6. *Vertex x_0 is adjacent to both vertices in X_3. Moreover, it has 2 neighbors in X_1 and 28 neighbors in X_2.*

Proof. For the sake of contradiction, suppose x_0 is adjacent to $k \in \{0, 1\}$ vertices of X_3. Let t be the number of neighbors of x_0 in X_1. By double counting 2-paths from x_0 to H we obtain:

$$4 \cdot 16 = 3k + t + 2(32 - k - t)\,,$$

implies $k = t$. Without loss of generality suppose that x_1 is not adjacent to x_0. By counting the number of 2-paths in a similar way we obtain that x_1 has 8 neighbors in X_2. But by strong regularity, x_0 and x_1 must have 16 common neighbors which is not possible since x_0, x_1 can share at most $k \leq 1$ common neighbors in X_1 and 8 common neighbors in X_2. Hence x_0 is adjacent to both x_1 and x_2 and so $k = t = 2$ and the claim follows. $\square$

In virtue of Lemma 4.2.6, let x_0', x_0'' be the vertices in X_1 that are adjacent to x_0.

Lemma 4.2.7. *Vertices x_1, x_2 each have 19 neighbors in X_1 and 9 neighbors in X_2. In particular, 12 or 13 vertices of X_1 are adjacent to both x_1 and x_2, 6 or 7 vertices only to x_1, and 6 or 7 only to x_2.*

Proof. The first part of the claim is an easy application of the already used double counting argument. Let now N_{x_1} and N_{x_2} be the neighbors of x_1 and x_2 in X_1 respectively. Since $|N_{x_1} \cup N_{x_2}| \leq 26$ (the size of X_1), we must have $|N_{x_1} \cap N_{x_2}| \geq 12$ by the inclusion-exclusion principle. On the other hand, x_1 and x_2 have 16 common neighbors. Since x_0 is a common neighbor and they have 2 or 3 common neighbors on H it follows that $|N_{x_1} \cap N_{x_2}| \leq 13$. The other assertions follow easily. $\square$

Lemma 4.2.8. *For $i = 1, 2$, the vertex x_i is adjacent to at least one of the vertices in $\{x_0', x_0''\}$.*

Proof. For $i = 1, 2$, the vertex x_i has 10 common neighbors with x_0. By Lemma 4.2.7, x_i only has 9 vertices in X_2, thus it must be adjacent to at least one of x_0', x_0''. $\square$

Let X_2^{-0} be the set of vertices in X_2 that are not adjacent to x_0. By Lemma 4.2.6, $|X_2^{-0}| = 14$.

Lemma 4.2.9. *At most one vertex from X_2^{-0} is adjacent to x_1, and at most one is adjacent to x_2.*

Proof. Vertex x_1 shares at most 2 common neighbors with x_0 in X_1 (possibly x_0' or x_0''). Thus it must have at least 8 out of 9 neighbors in X_2 adjacent to x_0. By symmetry, the claim holds for x_2. $\square$

Lemma 4.2.10. *Each vertex in $X_{75}[X_2^{-0}]$ that is not adjacent to any of the vertices in $\{x_1, x_2\}$ has degree $t \leq 2$ and it has t neighbors in $\{x_0', x_0''\}$. Vertices (at most two) in X_2^{-0} that are adjacent to exactly one of x_1 and x_2 have degree $t - 1$ in $X_{75}[X_2^{-0}]$ and $t \geq 1$ neighbors in $\{x_0', x_0''\}$. If there exists a vertex in X_2^{-0} that is adjacent to both in x_1 and x_2, then it is adjacent to both vertices in $\{x_0', x_0''\}$ and has degree 0 in $X_{75}[X_2^{-0}]$.*

Proof. Let $v \in X_2^{-0}$. Notice that v must have 16 common neighbors with x_0. First, assume it is not adjacent to x_1 or x_2. Then their common neighbors can only be in $\{x_0', x_0''\}$, say t of them, and in $X_2 \setminus X_2^{-0}$. By double counting 2-paths from v to H we obtain that v has 16 neighbors in X_2. Thus t of them must be in X_2^{-0}.

Second, assume that v is adjacent to exactly one of the x_1, x_2 (in this case it can be adjacent to only one of them). Then it has $16 - 1 - t$ neighbors in $X_2 \setminus X_2^{-0}$. On the other hand, by double counting, its degree in $X_{75}[X_2]$ is 14. Thus it's degree in $X_{75}[X_2^{-0}]$ is $t - 1$.

Finally, if v is adjacent to x_1 and x_2, it has degree 12 in $X_{75}[X_2]$, thus all these neighbors have to be in $X_2 \setminus X_2^{-0}$ and it also has to be adjacent to both vertices in $\{x_0', x_0''\}$. $\square$

Lemma 4.2.11. *Each of the vertices x_0', x_0'' has $15 - t$ neighbors in X_2^{-0}, where $t \in \{1, 2\}$ is the number of its neighbors in $\{x_1, x_2\}$. Moreover, x_0' and x_0'' are not adjacent.*

Proof. By double counting 2-paths from x_0' to H we have that x_0 has $25 - 2t$ neighbors in X_2. Vertices x_0 and x_0' have 10 common neighbors. Let s be equal to 1 if x_0' and x_0'' are adjacent and 0 otherwise. Vertices x_0 and x_0' must have $10 - t - s$ common neighbors in X_2, thus x_0' has $24 - 2t - (10 - t - s) = 15 - t + s$ neighbors in X_2^{-0}. Similar holds for x_0'' and since $|X_2^{-0}| = 14$, x_0' and x_0'' have more than 10 common neighbors. Thus they are not adjacent and $s = 0$. The lemma holds. □

The above lemmas give enough structure to be able to computationally obtain a small list of graphs that interlace X_{75} and must be induced subgraph of X_{75}, provided that X_{75} contains a H with this configuration.

Proposition 4.2.12. *There are 3597 graphs of the form $H \cup \{x_0, x_0', x_0'', x_1, x_2\} \cup X_2^{-0}$ that interlace X_{75}.*

4.2.3 Case $(2, 23, 45, 1)$

Let $X_0 = \{x_0, x_1\}, X_1, X_2, X_3 = \{x_3\}$ be the sets of vertices having $0, 1, 2$, and 3 neighbors in H respectively. Again, we start by proving certain structural claims about this configuration.

Lemma 4.2.13. $x_0 \not\sim x_1$.

Proof. If $x_0 \sim x_1$ then the number of 2-paths from x_0 to H is at most $3 + 2 \cdot 30$. But since x_0 is not adjacent to any vertex of H, it should have precisely $4 \cdot 16$ 2-paths to it. □

Lemma 4.2.14. $x_0 \sim x_3$ *and* $x_1 \sim x_3$.

Proof. Suppose x_0 is not adjacent to x_3 and let N_0, N_1, respectively, be the sets of neighbors of x_0, x_1 in X_2. By double counting 2-paths to H from x_0 or x_1 we have $|N_0| = 32$ while $|N_1| = 32 - 2t$ where $t \in \{0, 1\}$ depending on whether x_1 is adjacent to x_3 or not. Since all the neighbors of x_0 are in X_2, we have $|N_0 \cap N_1| = 16$. But this implies $|N_0 \cup N_1| = 48 - 2t \geq 46$ which is not possible as X_2 has size 45. □

Lemma 4.2.15. *Vertices x_0 and x_1 each have precisely one neighbor in X_1. In particular, these two neighbors are distinct.*

Proof. Let t be the number of neighbors of x_0 in X_1. By counting 2-paths from x_0 to H we have

$$16 \cdot 4 = 3 + t + 2 \cdot (32 - 1 - t)\,,$$

which gives that $t = 1$, and x_0 has 30 neighbors in X_2. Same holds for x_1. Let again N_0, N_1 be the sets of neighbors of x_0, x_1 in X_2. Then $|N_0 \cup N_1| \leq 45$, thus $|N_0 \cap N_1| \geq 15$. Since x_0 and x_1 have a common neighbor x_3, $|N_0 \cap N_1| = 15$ and $|N_0 \cup N_1| = 45$. This implies also that x_0 and x_1 cannot have a common neighbor in X_1. □

The last two lemmas now imply that X_2 can be partitioned into sets $X_2^0, X_2^{\{0,1\}}, X_2^1$ each of size 15 such that every vertex in X_2^i is adjacent to x_i and not adjacent to x_{1-i} for $i = 0, 1$ and every vertex in $X_2^{0,1}$ is adjacent to both x_0 and x_1. Let us denote the neighbors of x_0, x_1 in X_1 by x_0' and x_1' respectively.

Lemma 4.2.16. *If x_3 is adjacent to x_1' then it has 1 neighbor in X_2^0, otherwise it has no neighbor in X_2^0.*

Proof. The vertex x_3 has 10 neighbors in X_2 by double counting of 2-paths to H. On the other hand it must have 10 common neighbors with x_1. Notice that the common neighbors can only be in $X_2 \cup \{x_1'\}$. If x_3 is adjacent to x_1', then it must have 9 neighbors in $X_2^{0,1} \cup X_2^1$ thus 1 in X_2^0. On the other hand, if x_3 is not adjacent to x_1', then it must have all 10 neighbors in $X_2^{0,1} \cup X_2^1$ and no neighbor in X_2^0. □

Lemma 4.2.17. *The graph $X_{75}[X_2^0]$ has maximal degree 2. If $v \in X_2^0$ has degree 2, then it is not adjacent to x_3 but it is with x_1', if it has degree 1, it is adjacent with either both x_3 and x_1' or none of them, and if it has degree 0, it is adjacent to x_3 and not adjacent with x_1'.*

Proof. Pick a vertex $v \in X_2^0$ and let $s \in \{0,1\}$ indicate if it is adjacent with x_1' and $t \in \{0,1\}$ indicate if it is adjacent with x_3. Let r be the number of neighbors of v in X_2. We count 2-paths from v to H:

$$2 \cdot 10 + 2 \cdot 16 = 2 \cdot 3 + 3t + 2r + (32 - 2 - t - r - 1)\,.$$

Thus $r = 17 - 2t$. Vertices v and x_1 must have 16 common neighbors. This implies that the number of neighbors of v in $X_2^{0,1} \cup X_2^1$ is $16 - t - s$. Hence we have that v has $(17 - 2t) - (16 - t - s) = 1 - t + s$ neighbors in X_2^0 and hence the lemma follows. □

By generating graphs with the established structure we infer that none of them interlaces X_{75}.

Proposition 4.2.18. *No graph of the form $X_2^0 \cup \{x_0, x_1, x_1', x_3\} \cup H$ interlaces X_{75}.*

4.2.4 Case $(0, 29, 39, 3)$

Let X_1, X_2 and $X_3 = \{x_0, x_1, x_2\}$ be the respective subsets of vertices of X_{75}. There are three non-isomorphic ways to introduce 3 vertices to H by joining each vertex to 3 vertex of H. Each such graph G_1, G_2, G_3 can be uniquely described by a tuple $\vec{n} = (n_1, n_2, n_3, n_4)$ counting the number of edges from the i'th vertex of H to X_3. By relabeling the vertices of H if needed we obtain three tuples $(0, 3, 3, 3), (1, 3, 3, 2)$ and $(2, 2, 2, 3)$. We proceed by establishing certain structural claims about this configuration.

Lemma 4.2.19. *The vertices of X_3 form an independent set.*

Proof. No matter how we introduce edges among the vertices of X_3 in the graph $H \cup X_3$ we do not obtain a graph interlacing X_{75}. □

Lemma 4.2.20. *Every vertex $x \in \{x_0, x_1, x_2\}$ has 21 neighbors in X_1 and 8 neighbors in X_2.*

Proof. Let x have k neighbors in X_1 and $\ell = 32 - 3 - k$ neighbors in X_2. We count the number of paths of length 2 from x to H. We have $3 \cdot 10 + 16 = 3 \cdot 3 + k + 2 \cdot (32 - k - 3)$ and thus $k = 21$ and $\ell = 8$. □

Let X_2^0, X_2^1, X_2^3 be the neighbors in X_2 of x_0, x_1, x_2 respectively. We have proved that $|X_2^0| = |X_2^1| = |X_2^2| = 8$. Notice that the sets X_2^0, X_2^1, X_2^2 need not be disjoint.

Lemma 4.2.21. *It holds $|X_2^0 \cap X_2^1|, |X_2^0 \cap X_2^2|, |X_2^1 \cap X_2^2| \in \{0, 1\}$.*

Proof. Vertices x_0 and x_1 have 16 common neighbors, at least 2 of them are in H. Each of x_0, x_1 has 21 neighbors in X_1, where $|X_1| = 29$. Thus they must have at least 13 common neighbors in X_1. The latter implies that they have at most one common neighbor in X_2. Same holds for all other pairs from the assertion of the lemma. □

Lemma 4.2.22. *For $i \in \{1,2\}$ the graph $X_{75}[X_2^0 \setminus X_2^i]$ is triangle-free.*

Proof. Assume that there exists a triangle in $X_{75}[X_2^0 \setminus X_2^i]$. Together with x_0 it forms a 4-clique Q. Assume this Q is not contained in a 5-clique. Vertex x_i is not adjacent to any of the vertices of Q. Thus we have have a 4-clique with some vertices that are not adjacent to it. We have already covered this configuration before and shown that it is not feasible. On the other hand, if Q is contained in a K_5, we have a 5-clique and a vertex that is adjacent to at most one vertex on it. Again, this is not possible, since every vertex not on a 5-clique must have precisely two neighbors in it. □

Let X_1^{-0} denote the subgraph of X_1 induced on all the vertices not adjacent to x_0 and let X_1^i be the set of vertices in X_1 adjacent to x_i for $i \in \{0,1,2\}$.

Lemma 4.2.23. *The graph X_1^{-0} has 8 vertices. At most one of the vertices in X_1^{-0} is not adjacent to x_1 and at most one is not adjacent to x_2. Moreover, the graph on $X_1^{-0} \setminus X_1^i$, for $i \in \{1,2\}$, has no triangles. Each vertex $v \in X_1^{-0}$ has degree k in X_1^{-0} at most 3 and is adjacent to precisely $9-t-m+k$ vertices in X_2^0, where $t \in \{0,1,2\}$ is the number of vertices adjacent to v in $\{x_1, x_2\}$ and $m \in \{0,1\}$ the number of common neighbors of v and x_0 on H.*

Proof. By Lemma 4.2.20, $|X_1^0| = 21$, thus $|X_1^{-0}| = 8$. For $i \in \{1,2\}$, x_i and x_0 share at least 2 neighbors on H. Since they are non-adjacent, they share 16 neighbors, thus at most 14 in X_1. This implies $|X_1^0 \cap X_1^i| \leq 14$, thus $|X_1^0 \cup X_1^i| \geq 21 + 21 - 14 = 28$. Since $|X_1| = 29$ we see that there exist at most one vertex in X_1 that is not adjacent to x_0 and to x_i.

If there is a triangle in the graph induced by $X_1^{-0} \setminus X_1^i$, this triangle forms a H with x_0, while x_i is not adjacent to any of its vertices. If this 4-clique is not a part of a 5-clique the assertion follows since this case has already been dealt with (a H with some vertices not adjacent to it). On the other hand, if this H is a part of a K_5, we have an induced subgraph of K_5 together with a vertex that is adjacent to one or none of the vertices on K_5. Again, this is not possible since every vertex not on K_5 has precisely two neighbors in $V(K_5)$. Hence $X_1^{0,2}$ is indeed triangle-free.

Now let $v \in X_1^{-0}$ be as in the lemma. Denote with j the number of its neighbors in X_1. By counting 2-paths to H we get

$$10 + 3 \cdot 16 = 3 + 3t + j + 2(32 - 1 - t - j),$$

hence $j = 7 + t$. Denote with ℓ the number of neighbors of v in X_2^0. Vertex v and x_0 have 16 common neighbors, thus

$$16 = (j - k) + \ell + m = (7 + t - k) + \ell + m,$$

from which we get $9 - t - m + k = \ell \leq 8$. This implies also that $k \leq 2$. □

Lemma 4.2.24. *Let v be a vertex of $X_{75}[X_2^0]$ with degree k, $t \in \{0,1,2\}$ neighbors in $\{x_1, x_2\}$, and $m \in \{1,2\}$ the number of common neighbors of v and x_0 on H. Then:*

$$k + m + t \leq 3.$$

In particular $k \leq 2$.

Proof. Let $v \in X_2^0$. Denote with j the number of neighbors of v in X_1. By counting 2-paths from v to H we get:

$$2 \cdot 10 + 2 \cdot 16 = 2 \cdot 3 + 1 \cdot 3 + 3t + j + 2(32 - 2 - 1 - t - j),$$

thus $j = 15 + t$. Now let $\ell \leq 8$ be the number of neighbors of v in X_1^{-0}. Vertices v and x_0 have 10 common neighbors:

$$10 = k + m + (j - \ell) = k + m + (15 + t - \ell),$$

thus $5 + k + m + t = \ell \leq 8$ and $k + m + t \leq 3$. Since $m \in \{1, 2\}$, $k \leq 2$. □

By generating all graphs induced by $H \cup \{x_0, x_1, x_2\} \cup X_2^0 \cup X_1^{-0}$ we obtain

Proposition 4.2.25. *There are* 18089 *non-isomorphic graphs of the form* $H \cup \{x_0, x_1, x_2\} \cup X_2^0 \cup X_1^{-0}$ *that interlace* X_{75}.

The case analysis carried in this section resulted in a list of 21686 graphs, resulting in 6688644 star complements and roughly 40000 comparability graphs. By verifying that they all have clique number smaller than 56 we established Proposition 4.2.1.

Claim	Program	Output
Lemma 4.2.5	K4/126422/Claim-1.sage	/
Proposition 4.2.12	K4/126422/Case126422.sage	K4/12622/cands126422.g6
Proposition 4.2.18	K4/223451/Case223451.sage	/
Lemma 4.2.19	K4/029393/Claim1.sage	/
Proposition 4.2.25	K4/029393/generateFinal.sage	K4/029393/cands029393.g6
Lemma 4.3.3	K5/triangles/extendTriangle.sage	K5/triangles/candsTriag.g6

Table 4.1: Sage programs constructing small induced structure.

4.3 Main result

Let K_5 be a 5-clique of X_{75} with vertex set $\{k_1, \ldots, k_5\}$. The only possible configuration of vertices not in K_5 is $(0, 0, 70, 0, 0, 0)$, therefore every vertex of X_{75} that is not in K_5 has precisely two neighbors in K_5. For $1 \leq i < j \leq 5$, let $X_{i,j}$ be vertices in $V(X_{75}) \backslash V(K_5)$ that are adjacent to k_i and k_j. Since k_i and k_j are adjacent, they must have 10 common neighbors, 3 of them already on K_5. Hence $V(G) \setminus V(K_5)$ is partitioned into 10 sets of 7 vertices, namely $X_{0,1}, X_{0,2}, \ldots, X_{4,5}$. In what follows we establish structural results about these partitions.

Lemma 4.3.1. *For any* $1 \leq i < j \leq 5$ *the graph* $X_{i,j}$ *is either* $\overline{K_7}$ *or* $K_3 \cup \overline{K_4}$ *or* $K_1 \cup K_3 \cup K_3$.

Proof. Assume there exists an edge $e = \{x, y\}$ in the graph $X_{i,j}$. Then the vertices $\{x, y, k_i, k_j\}$ induce a 4-clique. By the result of the previous section, every 4-clique is contained in a 5-clique. Clearly, the additional vertex must be in $X_{i,j}$. Hence we have proved that every edge e in $X_{i,j}$ is contained in a triangle in $X_{i,j}$. Let T be a triangle in $X_{i,j}$ and $v \in X_{i,j}$ a vertex not on T. Since $T \cup \{k_i, k_j\}$ induces a 5-clique, every vertex not on this 5-clique is adjacent to exactly 2 vertices on this clique. Since v is adjacent to k_i and k_j, it is not adjacent to T and the lemma follows. □

As it turns out every pair of triangles in distinct partitions $X_{i,j}, X_{k,\ell}$ induce quite a regular structure.

Lemma 4.3.2. *Let $1 \leq i < j \leq 5$, $1 \leq k < \ell \leq 5$ and let T, T' be two triangles of $X_{i,j}$ and $X_{k,\ell}$, respectively. Let $c = |\{i, j, k, \ell\}|$. If $c = 3$, then the edges from T to T' form a perfect matching. If $c = 4$, they form a complement of a perfect matching.*

Proof. First assume $c = 3$. Since $T \cup \{k_i, k_j\}$ forms a 5-clique, every vertex of T' is adjacent to exactly 2 vertices in this 5-clique. Since $c = 3$, it must be adjacent to exactly one vertex in T. Similarly, every vertex of T must be adjacent to exactly one vertex in T'. Thus, the edges from T to T' form a perfect matching. The case when $c = 4$ is similar. $\square$

Our next lemma shows that not all partitions $X_{i,j}$ contain a triangle. In fact at most 7 do.

Lemma 4.3.3. *There are at least three distinct pairs $\{i, j\}, \{k, \ell\}, \{m, n\}$ such that $X_{i,j}, X_{k,\ell}$ and $X_{m,n}$ are independent sets of X_{75}.*

Proof. The proof uses a two stage pruning. Assuming at most two of the sets $X_{i,j}$, $1 \leq i < j \leq 5$, are independent, then at least 8 of these contain triangles. There exist two non-isomorphic ways to choose exactly eight sets among $X_{i,j}$, $1 \leq i < j \leq 5$, which contain triangles. By Lemma 4.3.2 if $X_{i,j}$ contains a triangle then it is isomorphic either to $K_3 \cup \overline{K_4}$ or $K_1 \cup K_3 \cup K_3$. Lemma 4.3.1 forces a structure of edge-sets between such pairs of triangles.

We have written a Sage program to construct all possibilities pruning out configurations that do not interlace X_{75}. Finally 117 non-isomorphic comparability graphs emerged starting from a single configuration. None of them contains a clique on 56 vertices and thus the lemma follows. $\square$

We are now able to prove our main theorem. The lists of graphs obtained in this part are too large to be hosted online hence they are not included in Table 4.1. However they can be obtained by a request to the authors.

Theorem 4.3.4. *The graph X_{75} does not exist.*

Proof. By the previous lemma, at least 3 graphs among $X_{i,j}$, for $1 \leq i < j \leq 5$, are independent sets. It is an easy check that there are 4 non-isomorphic configurations for the choice of 3 sets among $X_{i,j}$. These are: $(X_{1,2}, X_{2,3}, X_{4,5}), (X_{1,2}, X_{2,3}, X_{1,3}), (X_{1,2}, X_{2,3}, X_{2,4}), (X_{1,2}, X_{2,3}, X_{3,4})$. Notice that in all combinations we have sets $X_{1,2}, X_{2,3}$.

First we analyze the possible candidates for graphs induced by $\{k_1, \dots, k_5\} \cup X_{1,2} \cup X_{2,3}$. We do this by generating all bipartite graphs that can represent $X_{1,2} \cup X_{2,3}$. This is done using McKay's program *genbg*. Adding the vertices $\{k_1, \dots, k_5\}$ and removing non-interlacing graphs we end up with a list of 654325 graphs. By computing $\mathrm{sc}(G)$ for every such graph G and extending it to have order 19 we end up with a list of 361547477 star complements. By computing their respective comparability graphs and removing isomorphisms we end up with about 100^6 graphs. We have verified that none of these graphs has a clique of order 56 hence implying our assertion. $\square$

4.4 Computational aspects

In this section we briefly describe the computational tools and resources used to produce our result. As described in Section 4.1.1, our approach required generating a list of candidates for an induced subgraph of X_{75}, compute their comparability graphs and check their clique numbers. Most programs were written and tested independently in C and Sage, however most of the computation was performed only by C programs due to their efficiency.

4.4.1 Extending graphs and computing comparability graphs

As described in Section 4.1.1 we generated a list of graphs $\mathcal{L}$ such that if X_{75} exists then one of the graphs in $\mathcal{L}$ must be an induced subgraph of X_{75}. In order to rule out the existence of X_{75} we had to obtain star complements for each of the graphs in $\mathcal{L}$ and check the clique number of its comparability graphs. Some of the graphs in $\mathcal{L}$ already had star complements as induced subgraphs and were easy to handle. However some of the graphs did not, and in this case we had to find maximal induced subgraphs not having 2 as their eigenvalue, and extend them to have order 19. When choosing these subgraphs we tried to maximize the order of the automorphism group while minimizing the number of obtained subgraphs—note that two non isomorphic members of $\mathcal{L}$ may have isomorphic subgraphs. Both lists $\mathcal{L}$ and the one obtained from it are available on the GitHub page.

Let us remark that the process of extending graphs is computationally feasible whenever the obtained subgraphs have order 17. For otherwise we obtained far too many graphs of order 19.

The task of extending graphs to have order 19 was done by the already introduced program *extend.c* which takes as input a file with graphs given in graph6 string format and for each graph outputs all possible ways to introduce a new vertex to it so that the newly obtained graph interlaces X_{75}. If the input graph has a graceful pair of vertices then the extensions giving the minimal amount of graphs are written. Again, graphs are written in graph6 format.

After each iteration of *extend.c* we used McKay's *shortg* [58] program to remove isomorphic graphs from the obtained lists. Extending a graph of order 18 takes roughly 0.5 seconds on a standard desktop machine and the whole computation for the proof of our main result took roughly 240 CPU hours.

To compute comparability graphs we wrote a program that takes as input a list of star complements and for each output writes the graph6 representation of its comparability graph. The program is called *compGraph2graph6.c* and is found on GitHub [9]. We have found that the average comparability graph gets computed in 0.5 seconds and hence the instances of Theorem 4.3.4 were computed in about 5000 CPU hours. Let us note that the program does not output comparability graphs with order smaller than the clique number sought - in our case 57.

The computationally most intensive part was computing the clique number of the obtained comparability graph. This step took roughly 150000 CPU hours and was carried on a computational grid of 2000 CPU's. The computational grid that we used comprised of Intel Xeon CPUs clocked at 2670Mhz.

Both programs use the GNU GSL library for linear algebra routines and make use of the precision guaranteed by their implementation. Finally let us remark that in some of the steps we made use of the GNU parallel program [71].

4.4.2 Computing the clique number

While it is in general hard to compute the clique number of a graph, the structure of comparability graphs makes this task a little easier. As one may suspect by its definition, Comp offers a lot of symmetry which we exploit as follows. In what follows, $G[N(v)]$ will denote the subgraph of G induced by the neighbors of a vertex $v \in V(G)$.

Suppose we wish to compute the clique number $\omega(G)$ and let $v \in V(G)$. Then either v is contained in a maximal clique of G or is not. In the latter case the maximal clique of G equals the maximal clique in $\mathrm{Comp} - v$. In other words

$$\omega(G) = \max(\omega(G - v), \omega(G[N(v)])) + 1) \,.$$

Now, the key fact in computing $\omega(\mathrm{Comp})$ is that its automorphism group is fairly large and hence in computing its clique number we can remove the entire orbit $o(v)$ of a vertex v. Given that $o(v)$ is fairly large, the obtained graph $\mathrm{Comp} - o(v)$ is much smaller. We need not stop here. The key property that is used in the above idea is the fact that if $u, v \in V(G)$ are in the same orbit of $\mathrm{Aut}(G)$, then the graphs $G[N(u)]$ and $G[N(v)]$ are isomorphic. For our purposes we can define the *extended orbit of a graph G* as a partition of $\tilde{\mathcal{O}}(G)$ of $V(G)$ such that two vertices u, v are in the same part if and only if $G[N(u)] \cong G[N(v)]$. Summarizing the above idea into pseudo code we designed the following algorithm.

Algorithm 1 Algorithm for computing clique numbers of symmetric graphs

---

procedure CLIQUENUMBER(G,C)
  $cl \leftarrow 0$
  while $|V(G)| > c$ **do**
    $\tilde{\mathcal{O}} \leftarrow extendedOrbits(G)$
    if $|\tilde{\mathcal{O}}| = |V(G)|$ **then**
      break
    $o \leftarrow$ some orbit of $\tilde{\mathcal{O}}$
    $v \leftarrow$ an element of o
    $cltmp \leftarrow \mathrm{cliqueNumber}(G[v]), c) + 1$
    if $cltmp > cl$ **then**
      $cl \leftarrow cltmp$
    $G \leftarrow G - o$
  return $\max(cl, cliqueNumberBruteForce(G))$

In order to compute the clique number of our comparability graphs we used a variant of Algorithm 1 which leaves out two major details. Namely the computation of the extended orbits of G and the *cliqueNumberBruteforce* routine. For the latter, we needed an established program that calculates the clique number of a graph. We have found out that on our instances the program *mcqd* [53] drastically outperforms the well known clique finding algorithm *Cliquer* [62]. Hence whenever our input graph is small enough, we simply use *mcqd*. Since we only need to determine whether our graph has a clique of size at least 57 or not we made use of an additional optimization. Suppose we are trying to decide whether a graph G has a clique of size k and the greedy coloring algorithm shows that we can properly color the vertices of G using less than k colors. Then the clique number of G is smaller than k and we can stop our search. This is the essential idea behind the implementation of *mcqd* and we used it to obtain an even more efficient test for comparability graphs.

The second problem of computing the extended orbits is reduced to the problem of computing the orbits of the automorphism groups and canonical forms of graphs. While the computational complexity of these two problems is not settled, it is well known that in practice both problem offer efficient practical solutions. For example, it takes Bliss [48] about 5 seconds of CPU time, on our modest laptop, to compute the full automorphism group of a typical comparability graph of order 6000 and density 0.4.

In order to compute the extended orbits of a graph G we first compute the orbits $\mathcal{O}$ of its automorphism group. Finally for every representative of $\mathcal{O}$ we compute the canonical form of $G[N(v)]$ and join orbits with equal canonical forms.

A simple implementation of the above algorithm was implemented in Sage and is available on the GitHub [9] repository under the name *cliqueNumber.sage*.

Finally, let us remark that both *mcqd* and *Bliss* were integrated into Sage for the purposes of this thesis.

4.5 Final remarks

We have shown that a $(75, 32, 10, 16)$ SRG does not exists by presenting a classification approach based on the star complement technique. The main property that we exploited was the fact that such a SRG has an eigenvalue of high multiplicity, namely 56 which implies that the star complement graph has 19 vertices. Thus one can avoid the combinatorial explosion of constructing all possible star complements, provided that one can build large enough induced structure for the star complement graph. In our case this was established by building the star complement around a maximal clique of our SRG. Two things were crucial for our approach to work. First was the fact that many of the obtained comparability graphs were isomorphic thus significantly reducing the number of graphs whose clique number was to be determined. The second crucial part was the fact that comparability graphs had large automorphism groups thus allowing to exploit their symmetries when computing their clique number. We believe that a similar approach can be used to classify at least one of the following open parameters $(69, 20, 7, 5)$, $(95, 40, 12, 20)$, $(96, 45, 24, 18)$, $(99, 42, 21, 15)$.

Slovenski povzetek

Ta povzetek ima enako strukturo kot angleški del disertacije. Razdeljen je na 4 glavne dele - uvod, kromatični polinom, krepko regularni grafi in končno poglavje o $(75, 32, 10, 16)$ krepko regularnih grafih. Uvod služi kot orientacija v osnovne koncepte, ki jih srečamo v nadaljnjih poglavjih. V poglavju o kromatičnem polinomu navedemo nekaj osnovnih lastnosti tega objekta. Nadaljujemo z obravnavanjem vprašanja, ali obstaja graf, ki ni sebi komplementaren hkrati, pa ima enak kromatični polinom kot njegov komplement. Problem posplošimo tudi na Tuttov polinom. V poglavju o krepko regularnih grafih predstavimo nekaj osnovnih lastnosti takšnih grafov. Nadaljujemo z zanimivo povezavo med konveksnimi cikli in poddružino krepko regularnih grafov - Moorovi grafi. V zadnjem poglavju si podrobneje pogledamo strukturo $(75, 32, 10, 16)$ krepko regularnih grafov in pokažemo, da takšni grafi ne obstajajo.

Kazalo

Osnovi pojmi teorije grafov

Naj bo V končna množica. Množico vseh podmnožic V moči k označimo z $\binom{V}{k}$. Urejen par $G = (V, E)$, kjer $E \subseteq \binom{V}{2}$ je *graf*. Pravimo, da ima G množico *vozlišč* $V(G) = V$ ter množico *povezav* $E(G) = E$. Predstavljena definicija grafov je abstraktna, vendar obstaja mnogo aplikativnih scenarijev, v katerih nastopajo grafi. Grafe namreč srečamo pri modeliranju široke palete problemov v kemiji, sociologiji ter najpomembneje računalništvu. Ravno zato ni presenečenje, da je teorija grafov v zadnjih nekaj desetletjih doživela pravi razcvet. Bralce, ki jih zanima pregled razvoja teorije grafov, priporočamo vpogled v knjige [47], [20] in [29]. V nadaljevanju tega razdelka omenimo zgolj pojme in rezultate, ki so pomembni za našo disertacijo.

Če za dve vozlišči $x, y \in V(G)$ množica $\{x, y\}$ pripada $E(G)$ pravimo, da sta x in y *sosednji* in pišemo $x \sim_G y$. Če je graf G jasno določen iz konteksta, ga v notaciji za sosednost izpustimo. Število vozlišč, s katerimi je neko vozlišče $v \in V(G)$ sosedno, pravimo *stopnja* vozlišča v in jo označimo z $d(v)$. Če imajo vozlišča danega grafa enake stopnje, potem pravimo, da je graf *regularen z valenco* k, kjer je k stopnja poljubnega vozlišča. Primer regularnega grafa je tako imenovani *polni graf* na n vozliščih, K_n. Definiran je kot graf z množico vozlišč $V(K_n) = \{1, \ldots, n\}$ ter $\binom{V}{2}$ za množico povezav. Dejstvo, da smo za množico vozlišč polnega grafa vzeli $\{1, \ldots, n\}$, je nepomembno. Poljubna množica moči n bi dala strukturno enak graf. Slednji pojav modeliramo s pojmom *izomorfizma* grafov. Bodita G in H grafa ter $f : V(G) \to V(H)$ bijekcija, da za vsak par $x, y \in V(G)$ velja

$$x \sim_G y \iff f(x) \sim_H f(y)\,.$$

Takšni preslikavi f pravimo *izomorfizem* grafa. Za grafa G in H pa pravimo, da sta *izomorfna*. Izomorfizmu tipa $f : V(G) \to V(G)$ pravimo *avtomorfizem*. Množico vseh avtomorfizmov danega grafa G označimo z $\mathrm{Aut}(G)$. Ni se težko prepričati, da tvori $\mathrm{Aut}(G)$ grupo za operacijo kompozicije funkcij. Za dani graf G definiramo njegov *komplement* $\overline{G}$ kot graf z množico vozlišč $V(G)$ ter množico povezav $\binom{V(G)}{2} \setminus E(G)$. Grafu, ki je izomorfen svojemu komplementu, pravimo *sebi komplementaren* graf. Če je e povezava grafa G, potem z $G - e$ označimo graf z množico vozlišč $V(G)$ in z množico povezav $E(G) \setminus \{e\}$. Graf $G - v$ za vozlišče $v \in V(G)$ je definiran podobno.

Operacijo *skrčitve* dane povezave $e = \{x, y\}$ označimo z G/e in jo definiramo kot graf katerega množica vozlišč je $V(G - x)$ ter povezav $E(G - x) \cup \{\{x, x'\} \mid x' \neq x \wedge x' \sim_G y\}$. V disertaciji bomo srečali tudi pojem *grafa povezav*. Za dani graf G tvorimo graf povezav $L(G)$ tako, da za množico vozlišč vzamemo $E(G)$, dve vozlišči $L(G)$ pa sta sosednji natanko takrat, ko imata pripadajoči povezavi $E(G)$ skupno krajišče.

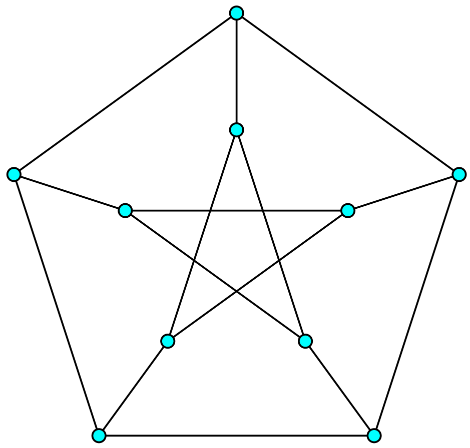
Primer grafa povezav predstavlja tako imenovani *Petersenov graf*, narisan na sliki 1. Z lahkoto se prepričamo, da je Petersenov graf $\overline{L(K_5)}$. Kot zadnjo omenjano znano operacijo *kartezičnega produkta*. Za grafa G in H označimo njun kartezični produkt z $G \square H$ in ga definiramo kot graf, katerega množica vozlišč je $V(G) \times V(H)$, soseščina med vozlišči pa je definirana s predpisom

$$(u, v) \sim_{G \square H} (u', v') \iff u = u' \wedge v \sim_H v' \quad \text{ali} \quad u \sim_G u' \wedge v = v'\,.$$

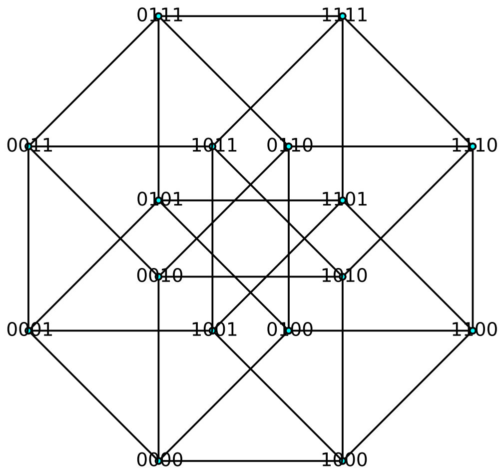
Primer 4.5.1. Klasičen primer kartezičnega produkta je graf hiperkocke dimenzije n definiran kot

$$Q_n = \square_{i=1}^n K_2\,.$$

Ekvivalentna definicija n-kocke pravi, da je Q_n graf katerega vozlišča so vsi binarni nizi dolžine n. Dve vozlišči (niza) pa sta sosednja natanko tedaj, ko se razlikujeta v natanko eni koordinati. Slika 2 ponazarja takšno konstrukcijo hiperocke dimenzije 4.



Slika 1: Najbolj znan objekt teorije grafov - Petersenov graf.



Slika 2: Hiperkocka dimenzije 4.

Za graf H pravimo, da je podgraf grafa G če $E(H) \subseteq E(G)$ in $V(H) \subseteq V(G)$. Pišemo $H \subseteq G$. Če za poljuben par $x, y \in V(H)$ velja, da $x \sim_G y \implies x \sim_H y$, potem pravimo, da je H *induciran* podgraf grafa G. Če $H \subseteq G$ in je H izomorfen polnemu graf, potem pravimo, da je H *klika* grafa G. Številu točk največje klike v grafu G imenujemo *klično število* in ga označimo z $\omega(G)$. Problem določanja kličnega števila je algoritmično težak problem. V razdelku 4.5 smo predstavili metodo za pohitritev računanja kličnega števila za bogat razred grafov.

Sprehod W v grafu G je zaporedje vozlišč $v_1, \dots, v_k$, ki zadošča pogoju, da za vsak $1 \leq i < k$ velja $v_i \sim v_{i+1}$. Če so vozlišča sprehoda različna, potem pravimo, da je W *pot* v grafu G. Če za W velja, da je $v_1 = v_k$, in so $v_2, \dots, v_{k-1}$ različna vozlišča, potem pravimo, da je W *cikel* dolžine $k + 1$. Grafu, kjer za vsak par vozlišč $u, v \in V(G)$ obstaja sprehod od u do v, pravimo *povezan* graf, z $d_G(u, v)$ pa označimo dolžino najkrajše poti med u in v. Z oznako $\mathrm{Diam}(G)$ označimo izraz $\max_{u,v \in V(G)} d_G(u, v)$ in ga imenujemo *premer* grafa G. Dolžini najkrajšega cikla grafa G pravimo *ožina* grafa. Če je G povezan 2-regularen graf z n vozlišči, ga imenujemo *cikel* dolžine n in ga označimo s C_n.

Barvanja grafov

Naj bo $c_k : V(G) \to \{1, \dots, k\}$ taka funkcija, da za poljubni dve vozlišči $x, y \in V(G)$ velja

$$x \sim y \Rightarrow c(x) \neq c(y)\,.$$

V takšnem primeru pravimo, da je *c pravilno* barvanje grafa G. Najmanjšemu številu k, ki garantira obstoj takšne funkcije, pravimo kromatično število grafa G in ga označimo z $\chi(G)$. Zveza $\chi(K_n) = n$ takoj da neenakost $\chi(G) \geq \omega(G)$. Kljub temu, da po definiciji $\chi(G) = k$ implicira obstoj pravilnega k barvanja grafa G, v splošnem takšno barvanje ni enolično. Naj $\widehat{c}_k$ označuje število pravilnih barvanj grafa G. Funkciji, ki šteje pravilna barvanja grafa G, pravimo *kromatični polinom.* Definiramo jo kot predpis $P_G : \mathbb{R} \to \mathbb{R}$ z lastnostjo, da je za vsako naravno število k zadoščena enakost

$$p_G(k) = \widehat{c}_k\,.$$

Osnovna trditev o kromatičnem polinomu vzpostavlja naslednjo lastnost.

Trditev 4.5.1. *Če je $e = \{x, y\}$ povezava grafa G, potem velja zveza $p_G(k) = p_{G-e}(k) - p_{G/e}(k)$.*

Ker je $p_{K_1}(k) = k$ ni težko videti, da trditev 4.5.1 upravičuje poimenovanje funkcije $p_G(x)$ kot polinom. V disertaciji predstavimo nekaj osnovnih lastnosti kromatičnega polinoma.

Grafovske matrike in spektralna teorija grafov

V tem odseku navajamo nekaj rezultatov iz spektralne teorije grafov, ki nam bodo omogočali lažje razumevanje nadaljnjih poglavjih. Če je G graf z množico vozlišč $\{v_1, \dots, v_n\}$, potem z A_G označimo njegovo matriko sosednosti, ki je definirana kot binarna $n \times n$ matrika $A_G = (a_{i,j})_{i,j=1}^n$ kjer je $a_{i,j} = 1$, če in samo če $v_i \sim v_j$.

Naj bo $w_{i,j}^k$ število sprehodov dolžine k med v_i in v_j. Naša prva trditev ne potrebuje nobenih dodatnih pojmov.

Trditev 4.5.2. *Ob upoštevanju zgornje notacije velja $A^k = (w_{i,j}^k)_{i,j=1}^n$.*

Na tem mestu naj omenimo, da trditev 4.5.2 ponuja učinkovit algoritem za računanje diametra danega grafa. Ob predpostavki, da je G povezan, je dovolj poiskati minimalen k tako, da so vsi elementi A^k neničelni. Izkaže se, da je velika verjetnost, da je pristop temeječ na tej ideji najbolj učinkovit algoritem za ta problem [66].

Matrika A_G je realna in simetrična, zato so njene lastne vrednosti realna števila. Z izrazom *lastne vrednosti grafa* G bomo imeli v mislih lastne vrednosti matrike A_G, ki jih bomo označili z

$$\lambda_1(G) \geq \cdots \geq \lambda_n(G)\,.$$

Graf G bomo iz opisane notacije spustili vsakič, ko bo G znan iz danega konteksta.

Veliko strukturnih lastnosti grafa G izhaja iz lastnih vrednosti matrike A_G. Prvi rezultat, ki ga predstavimo sega v leto 1962. Zanj ima zaslugo Harary [42]. Naj za podgraf H grafa G oznaka $c(H)$ šteje število povezanih komponent grafa H, ki so cikli. Podobno naj $r(H)$ označuje število povezanih grafa G, ki so izomorfni K_2. Naj bo $C_n(G)$ množica vseh vpetih podgrafov grafa G, katerih povezane komponente so izomorfne K_2 ali ciklu.

Izrek 4.5.3. Za vsak graf G velja

$$\det A_G = \sum_{H \in C_n(G)} (-1)^{r(H)} 2^{c(H)}\,.$$

Ker je $i$ti koeficient karakterističnega polinoma A_G vsota determinant vseh $i \times i$ glavnih podmatrik, takoj dobimo tudi naslednjo trditev.

Trditev 4.5.4. *Za i-ti koeficient c_i karakterističnega polinoma matrike A_G velja zveza*

$$(-1)^i c_i = \sum_{H \in C_i(G)} (-1)^{r(H)} 2^{c(H)}\,.$$

Kot je prvi opazil Sachs [68], nam zgornja trditev omogoča določiti liho ožino danega grafa ter tudi prešteti število najkrajših lihih ciklov.

Posledica 4.5.5. *Naj bo G graf lihe ožine $2r+1$ in*

$$p(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n,$$

karakteristični polinom A_G. Tedaj velja

$$c_3 = c_5 = \cdots = c_{2r-1} = 0\,.$$

Število $(2r+1)$-ciklov v G pa je

$$-c_{2r+1}/2\,.$$

Prepletanje

Za zaporedji realnih števil $\lambda_1 \geq \cdots \geq \lambda_n$ ter $\mu_1 \geq \cdots \geq \mu_m$, pri čemer je $n \geq m$, pravimo, da se *prepletata*, če je

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \quad \text{za} \quad i \in \{1, \ldots, m\}\,.$$

Znani izrek iz linearne algebre zagotavlja, da lastne vrednosti vsakega induciranega podgrafa prepletajo lastne vrednostni originalnega grafa [39]. Slednje je zelo uporabno dejstvo, ki je obrodilo veliko rezultatov v teoriji grafov.

Za naše namene bomo potrebovali močnejšo verzijo principa prepletanja, ki jo bomo formulirali na naslednji način. Naj bo $\mathcal{V} = (\mathcal{V}_1, \ldots, \mathcal{V}_k)$ razbitje $V(G)$. Naj bo $e(\mathcal{V}_i, \mathcal{V}_j)$ število povezav med vozlišči v V_i in vozlišči v V_j če $i \neq j$ sicer pa število povezav v množici V_i. Naj bo $A_{\mathcal{V}} = (a_{i,j})_{i,j=1}^k$ matrika definirana s predpisom

$$a_{i,j} = \begin{cases} \frac{e(\mathcal{V}_i, \mathcal{V}_j)}{|\mathcal{V}_i|} & \text{if} \quad i \neq j \\ \frac{2e(\mathcal{V}_i)}{|\mathcal{V}_i|} & \text{if} \quad i = j \end{cases} .$$

Izkaže se, da lastne vrednosti matrike $A_{\mathcal{V}}$ prepletajo lastne vrednosti grafa G, kar strnemo v naslednjo trditev.

Trditev 4.5.6. *Naj bo G graf in $\mathcal{V}$ razbitje njegovih vozlišč. Tedaj lastne vrednosti $A_{\mathcal{V}}$ prepletajo lastne vrednosti G.*

Kot bomo videli v nadaljevanju, nam princip prepletanja omogoča, da za dani graf H učinkovito določimo, ali je induciran podgraf v nekem večjem grafu G, katerega obstoja ne poznamo, poznamo pa njegove lastne vrednosti.

Kromatični polinom in Akiyama-Hararyjev problem

Pojem kromatičnega polinoma je prvi vpeljal Birkhoff [18] z željo, da bi mu slednje pomagalo rešiti problem štirih barv. V jeziku kromatičnega polinoma problem štirih barv pravi, da za vsak ravninski graf velja $p_G(4) > 0$. Sledne trditve Birkhoffu ni uspelo dokazati, vendar je neodvisno od tega pojem kromatičnega polinoma zaživel.

Študij kromatičnega polinoma se je skozi desetletja nadaljeval in v letu 1968 je Read objavil članek [65], v katerem je vzpostavil veliko osnovnih lastnosti kromatičnega polinoma. Ključen pojem vpeljan v omenjenem delu je bil pojem *kromatične ekvivalence*. Za grafa G in H pravimo, da sta kromatično ekvivalentna, če je $p_G(k) = p_H(k)$. V kontekstu te definicije pravimo, da je graf G *kromatično enoličen*, če je kromatično ekvivalenten zgolj z njemu izomorfnimi grafi. Primer takšnega grafa je npr. cikel. Določanje grafov, ki so kromatično enolični, je široko področje teorije grafov [51, 52].

V poznih sedemdesetih letih sta Akiyama in Harary objavila vrsto člankov [3, 4, 5, 6, 2, 8, 7, 1], ki je inicirala karakterizacijo grafov, ki se z njihovimi komplementi ujemajo v izbranih invariantah. Slednje je obrodilo področje teorije grafov, ki je aktivno vse do danes [59]. V sklopu njunih raziskovanj sta postavila tudi vprašanja in odprte probleme. Konkretno so v [7] vprašala, ali obstaja graf, ki ni sebi komplementaren, vendar ima enak kromatičen polinom kot njegov komplement. Odgovor na slednje vprašanje je pozitiven, dokaz pa sta podala Xu in Liu [75], ki sta skonstruirala neskončno družino takšnih grafov. Lastnost njune konstrukcije je, da imajo dobljeni grafi enako zaporedje stopenj vozlišč kot njihovi komplementi. V skladu s tem sta postavila domnevo

Domneva 1. Če ima graf G lastnost $p_G(k) = p_{\overline{G}}(k)$, potem imata G in $\overline{G}$ enako zaporedje stopenj vozlišč.

V naši disertaciji pokažemo, da velja naslednji rezultat.

Izrek 4.5.7. Obstaja neskončna družina grafov, ki imajo enak kromatičen polinom kot njihovi komplementi,vendar se razlikujejo v zaporedju stopenj vozlišč.

Pozornost nato usmerimo na posplošitev kromatičnega polinoma—Tuttov polinom $T_G(x,y)$. Za dano podmnožico $F \subseteq E(G)$ označimo z $c(F)$ število povezanih komponent grafa $(V(G), F)$. S to definicijo v mislih je Tuttov polinom definiran kot

$$T_G(x,y) = \sum_{F \subseteq E(G)} (x-1)^{c(F)-c(E)} \cdot (y-1)^{c(F)+|F|-|V(G)|} . \tag{4.3}$$

Da je slednje zares posplošitev omenjenega problema, vidimo iz znane zveze

$$p_G(k) = (-1)^{|V(G)|-k(E)} k^{c(E)} T_G(1-k, 0) .$$

V nadaljevanju disertacije pokažemo naslednji rezultat.

Izrek 4.5.8. Obstaja neskončna družina grafov, ki niso sebi komplementarni, vendar imajo enak Tuttov polinom kot njihovi komplementi.

Grafi iz predstavljene konstrukcije imajo lastnost, da se z osnovnim grafom ujemajo v zaporedju stopenj vozlišč, zato poglavje zaključimo z analognim problemom, ki je motiviral osnovno poglavje.

Problem 7. *Ali obstaja graf G, ki ima enak Tuttov polinom kot njegov komplement, vendar nima enakega zaporedja stopenj vozlišč?*

Krepko regularni grafi

Za k-regularen graf z v vozlišči pravimo, da je *krepko regularen* s parametri (v, k, λ, μ), če ima vsak par sosednjih vozlišč natanko λ skupnih sosedov, par nesosednih vozlišč pa natanko μ skupnih sosedov. Kot dodatno predpostavko zahtevamo, da so krepko regularni grafa diametra 2. Družina krepko regularnih grafov se pojavlja na veliko področjih teorije grafov, najbolj znana je njihova povezava z problemom izomorfizma ter njihova vloga v teoriji razdaljno regularnih grafov.

Primer 4.5.2. Najmanjši primer krepko regularnega grafa je cikel C_5, ki ima parametre $(5, 2, 0, 1)$. Naslednji naraven primer je Petersenov graf s parametri $(10, 3, 0, 1)$. Predstavljena grafa spadata v družino tako imenovanih *Moorovih grafov*. To so krepko regularni grafi s parametri $(k^2+1, k, 0, 1)$.

Primer 4.5.3. Za krepko regularen graf s parametri (v, k, λ, μ) se hitro pokaže, da je tudi $\overline{G}$ krepko regularen graf in sicer s parametri $(v, v-k-1, v-2-2k+\mu, v-2k+\lambda)$.

Primer 4.5.4. Kartezični produkt polnega grafa $K_n \Box K_n$ je prav tako krepko regularen graf s parametri $(n^2, 2n-2, n-2, 2)$.

V disertaciji razvijemo nekaj osnovnih lastnosti krepko regularnih grafov. Med drugim predstavimo njihovo ekstremalnost v smislu števila lastnih vrednosti. Znano je, da je edini graf, ki ima natanko dve različni lastni vrednosti poln graf K_n za $n \geq 2$. Izkaže se, da so krepko regularni grafi karakterizirani kot regularni grafi z natanko tremi različnimi lastnimi vrednostmi. Natančneje lahko za lastne vrednosti krepko regularnih grafov povemo naslednji rezultat.

Trditev 4.5.9. *Lastne vrednosti krepko regularnega grafa s parametri* (v, k, λ, μ) *so*

$$k,\ \frac{1}{2}\left[(\lambda-\mu) + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right] \quad \textit{in} \quad \frac{1}{2}\left[(\lambda-\mu) - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right],$$

pri čemer so

$$1,\ \frac{1}{2}\left[(v-1)-\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right] \quad \textit{in} \quad \frac{1}{2}\left[(v-1)+\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right]$$

ustrezne kratnosti.

S pomočjo trditve 4.5.9 z lahkoto izračunamo tudi lastne vrednosti razdaljne matrike krepko regularnega grafa. Velja:

Trditev 4.5.10. *Naj bo G krepko regularen graf s parametri (v,k,λ,μ), Število ν je lastna vrednost matrike A_G natanko tedaj, ko je $\frac{2}{\mu}\nu^2+(1-\frac{2\lambda}{\mu})\nu-\frac{2k}{\mu}$ lastna vrednost D_G.*

Naj bo $n_-(G), n_+(G)$ število negativnih oziroma pozitivnih lastnih vrednosti matrike D_G. V nadaljevanju disertacije pokažemo:

Trditev 4.5.11. *Obstaja neskončno mnogo povezanih grafov G, za katere je $n_+(G)>n_-(G)$.*

S slednjim smo odgovorili na vprašanje o obstoju takšnih grafov, postavljeno v [37]. V nadaljevanju disertacije se osredotočimo na povezavo med Moorovimi grafi in grafi z maksimalnim številom ciklov, katerih velikost je ožina grafa. Natančneje pokažemo:

Trditev 4.5.12. *Naj bo G graf z n vozlišči, m povezavami in liho ožino g. Število g-ciklov v G je največ $\frac{n}{g}(m-n+1)$, enakost pa je dosežena, če in samo če je G cikel ali Moorov graf.*

Zgornji rezultat naravno posplošimo v kontekst konveksnih ciklov. Za podgraf H grafa G pravimo da je *konveksen*, če za vsak par $x,y\in H$ velja, da je vsaka najkrajša pot med u in v v G vsebovana v H. Konveksnemu podgrafu, ki je cikel, pravimo *konveksni cikel*. S slednjo notacijo lahko zgornjo trditev posplošimo na naslednjo.

Trditev 4.5.13. *Naj bo G graf z n vozlišči m povezavami in ožino g. Število konveksnih ciklov v G je največ $\frac{n}{g}(m-n+1)$,, enakost pa je dosežena, če in samo če, je G Moorov graf ali cikel.*

Neobstoj $(75,32,10,16)$ krepko regularnega grafa

V tem razdelku opredelimo (ne)obstoj $(75,32,10,16)$ krepko regularnega grafa. Osnovna ideja predstavljenega pristopa temelji na metodi *zvezdnega komplementa*, ki sta jo razvila Cvetković in Rowlinson [26]. Naj bo G nek graf z lastno vrednostjo ψ, katere kratnost je k. Induciranemu podgrafu $H\subseteq G$ pravimo *zvezdni komplement* grafa G za lastno vrednost ψ, če ψ ni lastna vrednost H in ima H natanko $|V(G)|-k$ vozlišč. Izkaže se, da takšen graf vedno obstaja.

Trditev 4.5.14. *Naj bo G graf in ψ njegova lastna vrednost. Tedaj obstaja zvezdni komplement za G in ψ.*

Preden utemeljimo pomen zveznih komplementov, naj omenimo še naslednjo trditev, ki nam omogoča zgraditi zvezdni komplement iz danega, manjšega grafa.

Trditev 4.5.15. *Naj bo G graf z lastno vrednostjo ψ. Če je H' tak induciran podgraf grafa G, da H' nima lastne vrednosti ψ, potem obstaja nek zvezdni komplement H za G in ψ za katerega je H' induciran podgraf v H.*

Glavna lastnost zvezdnih komplementov leži v dejstvu, da lahko z nekim zvezdnim komplementom grafa G rekonstruiramo graf G. V nadaljevanju priredimo opisano trditev potrebam naše disertacije, bralca pa usmerimo na [26] za bolj podrobne implikacije te metode.

Naj bo H zvezdni komplement za G in ψ. Definirajmo produkt

$$\langle u, v\rangle = u(\psi I - A_H)^{-1} v^t\,.$$

Primerjalni graf grafa H in lastne vrednosti ψ je graf $\mathrm{Comp}(H, \psi)$ z vozlišči

$$V(\mathrm{Comp}(H,\psi)) = \{u \in \{0,1\}^{n-k} \mid \langle u, u\rangle = \psi \text{ and } \langle u, \overrightarrow{1}\rangle = -1\}$$

katerega sosednost je definirana s pravilom

$$u \sim v \iff \langle u, v\rangle \in \{-1, 0\}\,.$$

Naj v tej točki omenimo, da pogoj $\langle u, \overrightarrow{1}\rangle = -1$ izhaja iz dejstva, da je v našem kontekstu G regularen graf [67]. V splošnem omenjen pogoj ne velja.

Izkaže še, da se problem rekonstrukcije grafa G prevede na problem maksimalne klike v Comp. Specifično je za naš problem pomembno naslednje dejstvo.

Trditev 4.5.16. *Naj bo G nek graf z lastno vrednostjo ψ kratnosti k ter H ustrezen zvezdni komplement. Tedaj ima graf* $\mathrm{Comp}(H, \psi)$ *kliko velikosti k.*

Problem določanja obstoja $(75, 32, 10, 16)$ krepko regularnega grafa se torej prevede na naslednje. Po trditvi 4.5.9 vemo, da ima takšen graf lastno vrednost 2 kratnosti 56. Naj bo H nek graf z 19 vozlišči, ki nima 2 za lastno vrednost. Če je $\omega(\mathrm{Comp}(H, 2)) < 56$, potem H ni induciran podgraf v X. Problem določanja obstoja X se torej reducira na iskanje seznama grafov $\mathcal{L}$, tako da je vsaj en izmed elementov $\mathcal{L}$ induciran podgraf v X, hkrati pa so vsi grafi v $\mathcal{L}$ reda 19 in ne vsebujejo 2 za lastno vrednost. Če za vsak graf iz $\mathcal{L}$ pokažemo, da $\omega(\mathrm{Comp}) < 56$, potem X ne obstaja.

Ključnega pomena pri naslednjem pristopu je generiranje majhnega seznama $\mathcal{L}$ z opisano lastnostjo. V ta namen uporabimo princip prepletanja opisan v uvodnem poglavju. Drugi problem, ki se pojavi v tem kontekstu, je računanje velikosti maksimalne klike danega grafa G. Učinkovit algoritem za ta problem ni znan, hkrati pa so v praksi primerjalni grafi zelo veliki in gosti. Da bi se izognili omenjeni problematiki, je ključna uporaba simetrij grafov. Osnoven razveji in omeji pristopza iskanje maksimalne klike v grafu temelji na naslednji ideji. Če je $v \in V(G)$, potem bodisi v leži na neki maksimalni kliki ali v G ali ne. V slednjem primeru je maksimalna klika grafa G enaka maksimalni kliki v grafu $G - v$. Z drugimi besedami, za vsak graf G reda vsaj 2 velja

$$\omega(G) \quad = \quad \max(\omega(G - v), \omega(G[N(v)])) + 1)\,. \tag{4.4}$$

Če je $o(v)$ orbita vozlišča v glede na $\mathrm{Aut}(G)$, potem lahko enačbo (4.4) zapišemo kot

$$\omega(G) \quad = \quad \max(\omega(G - o), \omega(G[N(v)])) + 1)\,. \tag{4.5}$$

V primeru da ima G veliko simetrij, predstavlja to znatno redukcijo iskalnega prostora. To idejo lahko še posplošimo, saj je pogoj, ki ga orbite vozlišč predstavljajo ta, da je $G[N(v)] \cong G[N(u)]$ vsakič, ko pripadata u in v isti orbiti. Na osnovi te ideje smo zasnovali in implementirali algoritem *gtc* za iskanje klik. Slednji je bil ključen korak pri reševanju problema obstoja krepko regularnega grafa s parametri $(75, 32, 10, 16)$. S pomočjo omenjenega algoritma v disertaciji pokažemo, da:

Izrek 4.5.17. Krepko regularen graf s parametri $(75, 32, 10, 16)$ ne obstaja.

Kot posledico zgornjega izreka dobimo tudi

Izrek 4.5.18. Krepko regularen graf s parametri $(76, 35, 18, 14)$ ne obstaja.

Z orodji razvitimi v disertaciji verjamemo, da se bo dalo odgovoriti tudi na vprašanje obstoja nekaterih drugih krepko regularnih grafov.

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