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UNIVERSAL COMMUTATOR RELATIONS

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Adviser: prof. dr. Primož Moravec

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UNIVERZALNE KOMUTATORSKE RELACIJE

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vsaka opeka vsaka hiša vsaka mrvica vsak prah
vsako vlakno vsaka kri vsaka gora vsak sneg
vsako drevo vsako življenje vsaka dolina vsak prepad
vsako sovraštvo vsako jagnje vsako žarenje vsaka mavrica

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po vsem povsem mentorju
hvala

Abstract

We study relations between commutators in abstract groups. Of these, we expose universal relations that are consequences of purely algebraic manipulations. We focus on the commutator relations that are not of this sort. It is possible to collect the truly nontrivial nonuniversal commutator relations into an abelian group called the Bogomolov multiplier, which is the fundamental object of interest here. It is of particular interest to determine whether or not this object is trivial, or at least to have some control over it. The present thesis is an exposition of various aspects of this.

After presenting a brief historical motivation for studying this object, we explore some of its basic properties. Many examples of groups with both trivial and nontrivial Bogomolov multipliers are given, illustrating different techniques. We present a cohomological interpretation of the Bogomolov multiplier, which makes it possible to relate commutator relations to the study of commutativity preserving extensions of groups. The Bogomolov multiplier is a universal object parameterizing such extensions. A theory of covering groups is developed. These constructions are then used to produce an effective algorithm for computing Bogomolov multipliers of finite solvable groups. We further inspect groups that are minimal with respect to possessing nonuniversal commutator relations. The results of this are used to study the problem of triviality of the Bogomolov multiplier from the probabilistic point of view. We give an explicit lower bound for commuting probability that ensures triviality of the Bogomolov multiplier. Relative structural bounds on the Bogomolov multiplier are presented. By relating commuting probability to commutativity preserving extensions, these bounds are used to bound the Bogomolov multiplier relative to the commuting probability. We end by making use of another known apparition of the Bogomolov multiplier to give a negative answer to a conjecture of Isaacs about character degrees of finite groups arising from nilpotent associative algebras by adjoining a unit. The conjecture is tackled by considering such groups that arise from modular group rings. A more conceptual explanation for the observed irregular behavior is provided by looking at the situation from the point of view of algebraic groups. We show how elements of the Bogomolov multiplier can be seen as rational points on a certain commutator variety.

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Keywords: Bogomolov multiplier; commutativity preserving extension; commutator relation; commuting probability; fake degree conjecture.

Povzetek

Raziskujemo relacije med komutatorji v abstraktnih grupah. Med njimi izpostavimo univerzalne relacije, ki so zgolj posledice algebraičnih manipulacij. Osredotočimo se na komutatorske relacije, ki niso take. Te pristno netrivialne neuniverzalne komutatorske relacije je mogoče zbrati v abelovo grupo, imenovano multiplikator Bogomolova. Ta objekt ima tukaj osrednjo vlogo. Še posebej nas zanima vprašanje njegove trivialnosti in posedovanje nekakšnega nadzora nad njegovim obnašanjem. V disertaciji predstavimo razne vidike tega.

Po krajšem pregledu motivacije za študij multiplikatorja Bogomolova raziščemo nekaj njegovih osnovnih lastnosti. Podamo mnogo primerov grup s trivialnimi in netrivialnimi multiplikatorji Bogomolova ter prikažemo različne metode. Predstavimo kohomološko interpretacijo multiplikatorja Bogomolova, s čimer vzpostavimo povezavo med komutatorskimi relacijami in razširitvami grup, ki ohranjajo komutativnost. Multiplikator Bogomolova je univerzalen objekt, ki parametrizira takšne razširitve dane grupe. Razvijemo teorijo krovnih grup. Te konstrukcije uporabimo za izgradnjo učinkovitega algoritma za računanje multiplikatorjev Bogomolova končnih rešljivih grup. Nadalje raziščemo grupe, ki so minimalne glede na posedovanje neuniverzalnih komutatorskih relacij. Pridobljene rezultate uporabimo za študij problema trivialnosti multiplikatorja Bogomolova iz verjetnostnega vidika. Podamo eksplicitno spodnjo mejo za verjetnost komutiranja, ki zagotovi trivialnost multiplikatorja Bogomolova. Izpeljemo relativne strukturne meje v zvezi z multiplikatorjem Bogomolova. Verjetnost komutiranja povežemo z razširitvami, ki ohranjajo komutativnost, s čimer omejimo multiplikator Bogomolova v odvisnosti od verjetnosti komutiranja dane grupe. Nazadnje izkoristimo še eno znano pojavitev multiplikatorja Bogomolova, da podamo negativen odgovor na Isaacsovo domnevo o stopnjah karakterjev končnih grup, ki izhajajo iz nilpotentnih asociativnih algeber. Domnevi se približamo z grupami, ki izhajajo iz modularnih grupnih kolobarjev. Za opaženo neregularnost ponudimo tudi bolj konceptualno razlago z vidika algebraičnih grup. Pokažemo, da lahko elemente multiplikatorja Bogomolova vidimo kot racionalne točke na neki komutatorski raznoterosti.

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Contents

1

Introduction

This thesis is all about commutators, more or less in abstract groups. We take a group G, usually finite, and two of its elements $x, y \in G$. Their *commutator* is the group element $[x, y] = x^{-1}y^{-1}xy$, measuring the extent to which x and y do not commute. Since $[x, y]$ is a group element, the operation in the group can be exploited to compare different ways of not commuting. Given a set of commutators, we say that there is a *relation* between these commutators if some nontrivial product of them evaluates to be trivial in the group G.

There are some commutator relations that arise from the trivial reason of being a consequence of purely algebraic manipulations. Such is, for example, the relation $[x, y][y, x] = 1$ in G, which conveys no content about the structure of the group G. Such relations are called *universal commutator relations*. These can quickly be well understood, and so we focus on the commutator relations that are not of this sort. Additionally, we choose to ignore the simple commutators that evaluate into a trivial element of G, so as to isolate the truly nontrivial relations. It is possible to collect these nontrivial nonuniversal commutator relations into an abelian group called the *Bogomolov multiplier* of G, denoted by $\mathrm{B}_0(G)$.

The Bogomolov multiplier is the fundamental object of interest in this thesis. In the context of commutator relations, it was introduced recently by Moravec [Mor12], building on the work of Miller [Mil52] with a view towards Bogomolov's interpretation of unramified Brauer groups of quotient varieties [Bog87]. The Bogomolov multiplier represents an obstruction for the commutator relations of G to follow from the universal commutator relations while considering commuting pairs as redundant. It is therefore of particular interest to determine whether or not $\mathrm{B}_0(G)$ is trivial, or at least to have some control over it. The present thesis is an exposition of various aspects of this, based on [FAJ, GRJZJ, Jez14, JM14 GAP, JM14 128, JM15, JM].

Universal commutator relations

We begin by giving the basic definitions and present some motivation. Commutator relations are formally defined in Section 2.1. Of these, we expose universal relations and present Miller's Theorem 2.9 that gives a simple generating set for them. A measure of the extent to which commutator relations in a given group are consequences of the universal ones is brought forth in Section 2.2, where exterior squares are introduced. A connection with a certain homological object is thus established. In order to remove redundancies coming from commuting pairs in nonuniversal relations, we suitably factor the whole construction and henceforth focus on the group of nontrivial nonuniversal relations, i.e., the Bogomolov multiplier. A brief historical motivation for studying this object is presented in Section 2.3. It rests on finding counterexamples to Noether's approach to the rationality problem in algebraic geometry.

Basic properties and examples

In Section 3.1, some basic properties of the Bogomolov multiplier are explored. We show how Hopf's formula for integral homology can be adapted to give a description of the Bogomolov multiplier in terms of a free presentation of the given group. In some cases, this formula enables explicit calculations of the Bogomolov multiplier. We show that the objects under consideration match well with the notion of isoclinism, and further show that the functor B_0 is multiplicative. We also investigate its behavior with respect to taking quotients and subgroups. In both cases, exactness may be lost, and we inspect this in more detail. After establishing these basic properties, we turn to Section 3.2 to give many examples of groups with both trivial and nontrivial Bogomolov multipliers. Different techniques of doing this are illustrated. We consider groups with large abelian subgroups, symmetric groups, finite simple groups, Burnside groups, unitriangular groups, small p-groups, and p-groups of maximal nilpotency class. The latter case is particularly involved, and it is here that we first find natural examples of groups that possess many nontrivial nonuniversal relations. The result for maximal class then produces groups of arbitrary coclass with large Bogomolov multipliers. The coclass of a group of order p^n and nilpotency class c is the number $\bar{c} = n - c$.

Theorem (see Corollary 3.27 and Theorem 3.31)**.** *For every prime p, integers $\bar{c} \geq 1$ ($\bar{c} \geq 2$ for $p = 2$) and $C > 0$, there are infinitely many p-groups G of coclass $\bar{c}$ with $|\mathrm{B}_0(G)| > C$.*

Unraveling relations

The Bogomolov multiplier has a cohomological interpretation. It is therefore possible to think of commutator relations as extensions of groups. We begin developing the theory of such extensions in Section 4.1; they are characterized as being the extensions that preserve commutativity. More specifically, an extension of a group N by a group Q is a short exact sequence

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1 .$$

We say that this extension is commutativity preserving if commuting pairs of elements of Q have commuting lifts in G via π. Given a group Q and a Q-module N, we collect all commutativity preserving extensions of N by Q into a cohomological object

$\mathrm{H}^2_{\mathrm{CP}}(Q,N)$. Focusing on central extensions (those in which N is contained in the center of G), we show how the Bogomolov multiplier may be thought of as a universal object parameterizing such extensions of a given group via a version of the universal coefficient theorem.

Theorem (see Theorem 4.8)**.** *Let N be a trivial Q-module. Then there is a split exact sequence*

$$0 \longrightarrow \mathrm{Ext}(Q^{\mathrm{ab}},N) \longrightarrow \mathrm{H}^2_{\mathrm{CP}}(Q,N) \longrightarrow \mathrm{Hom}(\mathrm{B}_0(Q),N) \longrightarrow 0\,.$$

We also provide several other characterizations of these extensions in terms of the kernel of the extension. In Section 4.2, we prove that commutativity preserving extensions behave well with respect to isoclinism of extensions. A theory of covering groups is then developed. First of all, we reduce the situation to considering only extensions of N by Q with the property $N \leq Z(Q) \cap [Q,Q]$. Such extensions are termed to be stem central. A CP cover of Q is any stem central CP extension whose kernel is of the same order as $\mathrm{B}_0(Q)$. We prove that CP covers indeed possess a covering property.

Theorem (see Theorem 4.16)**.** *Let Q be a finite group given via a free presentation $Q = F/R$. Set*

$$H = \frac{F}{\langle \mathrm{K}(F) \cap R\rangle} \quad \textit{and} \quad A = \frac{R}{\langle \mathrm{K}(F) \cap R\rangle},$$

where $\mathrm{K}(F)$ is the set of commutators of F.

1. *The group A is a finitely generated central subgroup of H and its torsion subgroup is*
$$T(A) = \frac{[F,F] \cap R}{\langle K(F) \cap R\rangle} \cong \mathrm{B}_0(Q).$$
2. *Let C be a complement to $T(A)$ in A. Then H/C is a CP cover of Q. Every CP cover of Q can be obtained in this way.*
3. *Let G be a stem central CP extension of a group N by Q. Then G is a homomorphic image of H and in particular N is a homomorphic image of $\mathrm{B}_0(Q)$. Thus, CP covers of Q are precisely the stem central CP extensions of Q of maximal order.*

We show that the Bogomolov multiplier of a CP cover is trivial. In this way, it is possible to view commutator relations much as loops that can be unraveled by passing to a suitable cover.

Theorem (see Corollary 4.21)**.** *Let Q be a group and G a CP cover of Q. For every filtration of subgroups*

$$1 = N_0 \leq N_1 \leq \cdots \leq N_n = \mathrm{B}_0(Q),$$

there is a corresponding sequence of groups $G_i = G/N_i$, where G_i is a central CP extension of G_j with kernel $N_j/N_i \cong \mathrm{B}_0(G_j)/\,\mathrm{B}_0(G_i)$ whenever $i \leq j$.

This further enables a thorough inspection of both maximal and minimal commutativity preserving extensions.

Theorem (see Theorem 4.30)**.** *The group* $\mathrm{H}^2_{\mathrm{CP}}(Q, \mathbb{Z}/p\mathbb{Z})$ *is elementary abelian of rank* $\mathrm{d}(Q) + \mathrm{d}(\mathrm{B}_0(Q))$*.*

These constructions are then used in Section 4.3, where we present an effective algorithm for computing Bogomolov multipliers of finite solvable groups. The algorithm is able to recognize the commutator relations of the group that constitute its Bogomolov multiplier. As a sample case we use the algorithm to determine the multipliers of all groups of order 128. In Section 4.4, we inspect groups that are minimal with respect to possessing nonuniversal commutator relations. More specifically, a finite group G is termed to be a B_0-minimal group whenever $\mathrm{B}_0(G) \neq 0$ and for every proper subgroup H of G and every proper normal subgroup N of G, we have $\mathrm{B}_0(H) = \mathrm{B}_0(G/N) = 0$. These groups may be thought of as building blocks of groups with nontrivial Bogomolov multipliers. Some strong restrictions on the structure of B_0-minimal groups are found and we classify such groups of nilpotency class 2.

Commuting probability bounds

We consider the problem of triviality of B_0 from the probabilistic point of view. The structure of Bogomolov multipliers heavily depends on commuting pairs of elements of a given group. We therefore inspect the probability that a randomly chosen pair of elements of a given group G commute,

$$\mathrm{cp}(G) = \frac{|\{(x, y) \in G \times G \mid [x, y] = 1\}|}{|G|^2}.$$

This notion is introduced in Section 5.1, together with some of its basic properties. The probability of commuting may be thought of as a type of measure of how close the given group is to being abelian. In this sense, having $\mathrm{cp}(G)$ bounded away from zero ensures abelian-like properties of the group G. We explore this first in Section 5.2, where we give an explicit lower bound for commuting probability that ensures triviality of the Bogomolov multiplier. This is first done for p-groups.

Theorem (see Theorem 5.7)**.** *Let G be a finite p-group. If*

$$\mathrm{cp}(G) > \frac{2p^2 + p - 2}{p^5},$$

then $\mathrm{B}_0(G)$ *is trivial.*

It is then easy to obtain a global bound applicable to all finite groups.

Theorem (see Corollary 5.8)**.** *Let G be a finite group. If* $\mathrm{cp}(G) > 1/4$*, then* $\mathrm{B}_0(G)$ *is trivial.*

The proof of Theorem 5.7 is quite involved and is based on studying minimal counterexamples, which turn out to be B_0-minimal groups. Using this, we establish a nonprobabilistic criterion for the vanishing of the Bogomolov multiplier. We also use the absolute commuting probability bound to produce some curious examples of B_0-minimal groups of arbitrary large nilpotency class. These examples contradict parts of [Bog87, Theorem 4.6 and Lemma 5.4]. In Section 5.3, we relate commuting probability to commutativity preserving extensions.

Theorem (see Proposition 5.12)**.** *A central extension*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is a commutativity preserving extension if and only if $\mathrm{cp}(G) = \mathrm{cp}(Q)$*.*

This observation is then applied to show how the theory of CP covers can be used to produce structural bounds on the Bogomolov multiplier.

Theorem (see Proposition 5.16)**.** *Let Q be a finite group and S a normal subgroup such that Q/S is cyclic. Then* $|\mathrm{B}_0(Q)|$ *divides* $|\mathrm{B}_0(S)| \cdot |S^{\mathrm{ab}}|$*, and* $\mathrm{d}(\mathrm{B}_0(Q)) \leq \mathrm{d}(\mathrm{B}_0(S)) + \mathrm{d}(S^{\mathrm{ab}})$*.*

Theorem (see Proposition 5.17)**.** *Let Q be a finite group and S a subgroup. Then* $\mathrm{B}_0(Q)^{|Q:S|}$ *embeds into* $\mathrm{B}_0(S)$*.*

These are used to give a nonabsolute version of previous results and bound the Bogomolov multiplier relative to the commuting probability.

Theorem (see Theorem 5.20)**.** *Let $\epsilon > 0$, and let Q be a group with* $\mathrm{cp}(Q) > \epsilon$*. Then* $|\mathrm{B}_0(Q)|$ *can be bounded in terms of a function of* $\max\{\mathrm{d}(S) \mid S \text{ a Sylow subgroup of } Q\}$ *and ϵ. Moreover,* $\exp \mathrm{B}_0(Q)$ *can be bounded in terms of a function of ϵ.*

We end by exposing a curious corollary concerning the exponent of the Schur multiplier.

Theorem (see Corollary 5.21)**.** *Given $\epsilon > 0$, there exists a constant $C = C(\epsilon)$ such that for every group Q with* $\mathrm{cp}(Q) > \epsilon$*, we have* $\exp \mathrm{M}(Q) \leq C \cdot \exp Q$*.*

Rationality revisited

In this final chapter, we make use of another known apparition of the Bogomolov multiplier to give a negative answer to a conjecture of Isaacs about character degrees of certain groups. We begin Section 6.1 by first widening our context to Lie groups and consider their representations. The Kirillov orbit method is a well-known strategy of obtaining representations. We present the idea behind the method and give a more detailed description for the class of algebra groups. These are groups of the form $1 + A$ with A a nilpotent associative algebra over a field $\mathbb{F}$, which will typically be finite. We explore the orbit method for this class of groups on the most basic level, leading up to the so called Fake degree conjecture. This is further reduced to the following question.

Question (see Question 6.6). *Is it true that the size of the group abelianization of* $1+A$ *coincides with the size of the Lie abelianization of* A*?*

We tackle the conjecture in Section 6.2 by considering algebra groups that arise from modular group rings $\mathbb{F}_q[X]$ of finite p-groups X. In this case, it is easy to compute the Lie abelianization of the appropriate augmentation algebra $\mathrm{I}_{\mathbb{F}_q}$. What is substantially more difficult is to compute the size of the abelianization of the group of units $1+\mathrm{I}_{\mathbb{F}_q}$. By translating the problem into a K-theoretical one, the Bogomolov multiplier enters into play via Whitehead groups. Here, the Bogomolov multiplier represents an obstruction to having a well-behaved exp-log correspondence between a given algebra group and its Lie algebra. In the end, we are able to prove the following.

Theorem (see Theorem 6.10). *Let* X *be a finite* p*-group. Then*

$$|(1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}| = q^{\mathrm{k}(X)-1}|\mathrm{B}_0(X)|.$$

The Fake degree conjecture is thus refuted in all characteristics. Ending with Section 6.3, we provide a more conceptual explanation for this irregular behavior over finite fields. This is done by looking at the situation from the point of view of algebraic groups. Taking an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$, one can think of $\mathbf{G} = 1+\mathrm{I}_{\mathbb{F}}$ as an algebraic group defined over $\mathbb{F}_p$. We write $\mathbf{G}(\mathbb{F}_q)$ for the $\mathbb{F}_q$-points of $\mathbf{G}$. We show how elements of the Bogomolov multiplier can be seen as rational points on a certain commutator variety.

Theorem (see Theorem 6.14). *Let* X *be a finite* p*-group and* $\mathbf{G} = 1+\mathrm{I}_{\mathbb{F}}$.

1. *We have*
$$|\mathbf{G}(\mathbb{F}_q) : \mathbf{G}'(\mathbb{F}_q)| = q^{\mathrm{k}(X)-1}.$$
2. *For every* $q = p^n$*, we have*
$$\mathbf{G}'(\mathbb{F}_q)/\mathbf{G}(\mathbb{F}_q)' \cong \mathrm{B}_0(X).$$

This also gives a very fresh interpretation of the exponent of the Bogomolov multiplier.

Theorem (see Theorem 6.15). *Let* X *be a finite* p*-group and* $\mathbf{G} = 1+\mathrm{I}_{\mathbb{F}}$*. We have*

$$\exp \mathrm{B}_0(X) = \min\{m \mid \mathbf{G}'(\mathbb{F}_q) \subseteq \mathbf{G}(\mathbb{F}_{q^m})'\}.$$

2

Universal commutator relations

Commutator relations are introduced in a formal manner. Of these, we expose universal relations and find a simple generating set for them. Then a way to measure the extent to which other relations are consequences of the universal ones is put in place via exterior squares. This produces a connection with a certain homological object. After removing redundancies, we focus on the group of nontrivial nonuniversal relations, called the Bogomolov multiplier. This is the central object of the thesis. We briefly present its historical role in Noether's rationality problem.

This chapter is based on [Bog87, GS06, Mil52, Mor12].

2.1 Commutator relations

2.1.1 Relations between commutators

Let G be a group. Consider a set of its commutators

$$[x_1, y_1],\ [x_2, y_2],\ \ldots,\ [x_n, y_n] \tag{2.1}$$

for some $x_i, y_i \in G$ with $1 \leq i \leq n$. We say that there is a *relation* between these commutators if there exists a nontrivial word $\omega(z_1, z_2, \ldots, z_n)$ in the free group $F_n\langle z_1, z_2, \ldots, z_n\rangle$ on n generators such that

$$\omega([x_1, y_1], [x_2, y_2], \ldots, [x_n, y_n]) = 1$$

in the group G.

***Example* 2.1.** *Take G to be an abelian group. Pick any two elements $x, y \in G$. Then $[x, y] = 1$ and the relation is observed by the word $\omega(z_1) = z_1$ in $F_1\langle z_1\rangle$.*

***Example* 2.2.** *Take G to be any finite group. Pick any two elements $x, y \in G$. Then $[x, y]^{|G|} = 1$ and the relation is observed by the word $\omega(z_1) = z_1^{|G|}$ in $F_1\langle z_1\rangle$.*

***Example* 2.3.** *Take G to be any group. Pick any two elements $x, y \in G$. Then $[y, x][x, y] = 1$ and the relation is observed by the word $\omega(z_1, z_2) = z_1 z_2$ in $F_1\langle z_1, z_2\rangle$.*

***Example* 2.4.** *Take G to be any group. Pick any three elements $x, y, z \in G$. Then $[xy, z] = [x^y, z^y][y, z]$, or equivalently $[xy, z]^{-1}[x^y, z^y][y, z] = 1$, and the relation is observed by the word $\omega(z_1, z_2, z_3) = z_1^{-1}z_2z_3$ in $F_1\langle z_1, z_2, z_3\rangle$. There is an analogous relation corresponding to the expansion of the commutator $[x, yz]$.*

***Example* 2.5.** *Take G to be any group. Pick any three elements $x, y, z \in G$. Then*

$$[[x^y, y^{-1}], z^y][[y^z, z^{-1}], x^z][[z^x, x^{-1}], y^x] = 1$$

and the relation is observed by the word $\omega(z_1, z_2, z_3) = z_1z_2z_3$ in $F_1\langle z_1, z_2, z_3\rangle$.

It is evident that in order to inspect commutator relations, we need to provide a more detailed representation rather than solely considering them as words in commutators.

2.1.2 Identifying commutator relations

Suppose that there is a relation between the commutators (2.1) in the group G. We represent this relation by a word whose variables are not commutators but rather indeterminates representing each particular group commutator. To this end, consider the free group

$$(G, G) = \langle (x, y) \mid x, y \in G\rangle,$$

freely generated by the set $G \times G$. There is a natural epimorphism $\kappa\colon (G, G) \to G$ with $\kappa(x, y) = [x, y]$ for every $x, y \in G$. A *relation* between the commutators (2.1) is defined to be a word $\omega \in (G, G)$ with support in the set $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ such that $\kappa(w) = 1$ in G. The group of all commutator relations of G will be denoted by $\mathfrak{R}_G = \ker \kappa$. Note that G acts by component wise conjugation on (G, G). The subgroup $\mathfrak{R}_G$ is invariant under this action, and so G acts on $\mathfrak{R}_G$. Given elements $\omega_1, \omega_2 \in (G, G)$, we will often say and write that $\omega_1 = \omega_2$ is a relation, interpreting the equality in $(G, G)/\mathfrak{R}_G$ and meaning that $\omega_1\omega_2^{-1}$ is a relation.

***Example* 2.6.** *Take G be any group. Pick any two commuting elements $x, y \in G$. Then $[x, y] = 1$ and there is a relation $\omega = (x, y) \in \mathfrak{R}_G$.*

***Example* 2.7.** *Take G to be any group. Pick any two elements $x, y \in G$. Then $[y, x][x, y] = 1$ and there is a relation $\omega = (y, x)(x, y) \in \mathfrak{R}_G$.*

***Example* 2.8.** *Take G to be any group. Pick any three elements $x, y, z \in G$. Then $[xy, z] = [x^y, z^y][y, z]$ and there is a relation $\omega = (xy, z)^{-1} \cdot (x, z)^y(y, z) \in \mathfrak{R}_G$. There is an analogous relation corresponding to the expansion of the commutator $[x, yz]$.*

2.1.3 Universal relations and Miller's theorem

There are some commutator relations that arise from the trivial reason of being a consequence of purely algebraic manipulations. Such are the relations in Examples 2.7 and 2.8, where no knowledge about the group G is required. Observe that such relations in any group G are precisely the ones that are inherited from relations in a free group.

More specifically, let F be a free group together with a homomorphism $\varphi\colon F \to G$. Then there is an induced homomorphism $\varphi_{\mathfrak{R}}\colon \mathfrak{R}_F \to \mathfrak{R}_G$. In this way, one may transfer the relations $\mathfrak{R}_F$ to G. The relations $\mathfrak{R}_F$ are accordingly called *universal commutator relations*. This definition refers to any free group F. When considering a specific group G, the same name is used for the union of images of all possible homomorphisms $\varphi_{\mathfrak{R}}$ with F varying.

There is a clear description of a generating set for the universal commutator relations.

Theorem 2.9 ([Mil52])**.** *Let F be a free group. Then $\mathfrak{R}_F$ may be generated as a normal subgroup of (F,F) by the relations*

$$(xy,z)=(x,z)^y(y,z),\quad (x,yz)=(x,z)(x,y)^z,\quad (x,x)=1 \tag{2.2}$$

for all $x,y,z\in F$.

During the course of the proof, we will require some of the many consequences of the relations (2.2). Set $\mathfrak{S}_F$ to be the normal subgroup of (F,F) generated by the relations (2.2).

Lemma 2.10. *The following relations belong to $\mathfrak{S}_F$:*

$$(1,z)=1,\quad (x,y)^{-1}=(y,x),\quad (x^{-1},y)=(x,y)^{-x^{-1}},$$
$$(x,y)^z=(x,y)([x,y],z),\quad (x,y)^{(z,w)}=(x,y)^{[z,w]},\quad [(x,y),(z,w)]=([x,z],[y,w]),$$
$$(x,y)(z,w)=(z,[y,x])^{-1}(z,w[y,x])(x,y)$$

for all $x,y,z,w\in F$.

Proof. The first of these is obtained by inserting $x=y=1$ into (2.2). Next, by computing in $(F,F)/\mathfrak{S}_F$ we have

$$1=(xy,xy)=(xy,y)\cdot(xy,x)^y=(x,y)^y(y,y)\cdot(x,x)^{y^2}(y,x)^y$$

Replacing x by $x^{y^{-1}}$ yields $1=(x,y)(y,x)$ in $(F,F)/\mathfrak{S}_F$. The third relation can now be obtained by inserting $y=x^{-1}$ into (2.2). By computing in $(F,F)/\mathfrak{S}_F$ we have

$$(z,[x,y])=(z,y^{-x}y)=(z,y)(z,y^{-x})^y=(z,y)(z^{x^{-1}},y^{-1})^{xy}=(z,y)(xzx^{-1},y^{-1})^{xy}.$$

This can further be reduced into

$$(z,y)(xz,y^{-1})^y(x^{-1},y^{-1})^{xy}=(z,y)(xz,y)^{-1}(x,y)=(x,y)^{-z}(x,y),$$

as required. Next, expand (xz,yw) in two ways using the relations (2.2), first with respect to the first component and then with respect to the second, and vice-versa. Comparing the two, we obtain the relation

$$((z,w)(x,y)^{zw})^{-1}\,(x,y)^{wz}(z,w).$$

Replacing x and y by $x^{(wz)^{-1}}$ and $y^{(wz)^{-1}}$ then gives

$$(x,y)^{(z,w)}=(x,y)^{[z,w]}.$$

This immediately implies the next relation, and also the final one after expanding $(z,w[y,x])$ with respect to the second component. $\square$

Proof of Theorem 2.9. Consider the quotient group $\mathfrak{Q}_F = \mathfrak{R}_F/\mathfrak{S}_F$. Note that associating $\mathfrak{Q}_F$ to F is a functor. We claim that $\mathfrak{Q}_F$ is trivial.

It suffices to inspect only finitely generated groups F, as every commutator relation has finite support. In case F is cyclic, then $\mathfrak{Q}_F = 1$ by virtue of the element (x^n, x^m) belonging to $\mathfrak{S}_F$ for all integers n, m. The general case is then done by induction on the number of generators. Decompose F as a free product $F = F_1 * F_2$. Let $i_1 \colon F_1 \to F$ and $i_2 \colon F_2 \to F$ be the natural injections. The induced maps $i_{1\mathfrak{Q}}$ and $i_{2\mathfrak{Q}}$ are both trivial by assumption. We will now show that $\mathfrak{Q}_F$ is generated by the images $\operatorname{im} i_{1\mathfrak{Q}}$ and $\operatorname{im} i_{2\mathfrak{Q}}$, thus proving the theorem.

In order to do this, we will consider the subgroups $X_1 = i_{1\mathfrak{R}}(F_1, F_1)$, $X_2 = i_{2\mathfrak{R}}(F_2, F_2)$, $Y = \langle (f_1, f_2) \mid f_1 \in F_1, f_2 \in F_2 \rangle$ of the group (F, F). Take an element $(x, y) \in (F, F)$. Write x and y as a product of elements of F_1 and F_2 and use the first and second relations of (2.2) to see that modulo $\mathfrak{S}_F$ the element (x, y) may be written as a product of terms of the form $(f_1, f_1')^f$, $(f_2, f_2')^f$, $(f_1, f_2)^f$ and $(f_2, f_1)^f$ with $f_1, f_1' \in F_1$, $f_2, f_2' \in F_2$ and $f \in F$. Each element of this form may in turn be broken down into a product of terms of the same type without the exponent f appearing. This is achieved by repeated use of the following rules:

$$\begin{aligned}
(f_1, f_1')^{f_2} &= (f_1, f_1')([f_1, f_1]', f_2),\\
(f_1, f_2)^{f_1'} &= (f_1 f_1', f_2)(f_1', f_2)^{-1},\\
(f_1, f_2)^{f_2'} &= (f_1, f_2')^{-1}(f_1, f_2 f_2').
\end{aligned}$$

The first of these is contained in Lemma 2.10 and the other two are restatements of the defining relations (2.2). Thus we see that every element of (F, F) is congruent modulo $\mathfrak{S}_F$ to a product of positive powers of terms (f_1, f_1'), (f_1, f_2), (f_2, f_1) and (f_2, f_2').

Now, every term (f_2, f_2') can be moved to the far right via the last two relations in Lemma 2.10. Similarly, all terms (f_1, f_1') can be moved to the far left. Additionally, mixed terms (f_2, f_1) can be replaced by $(f_1, f_2)^{-1}$.

Finally, let $\omega \in \mathfrak{R}_F$ be a universal relation. Rewriting ω modulo $\mathfrak{S}_F$, we have $\omega = \alpha_1 \mu \alpha_2$, where α_1 is a product of terms (f_1, f_1'), α_2 is a product of terms (f_2, f_2') and μ is a product of terms (f_1, f_2). Projecting ω onto F_1, we see that $\alpha_1 \in \mathfrak{R}_{F_1}$. Similarly, $\alpha_2 \in \mathfrak{R}_{F_2}$. Thus we also have $\mu \in \mathfrak{R}_F$. The latter is, however, only possible when $\mu = 1$ in (F, F), as F is a free product of F_1 and F_2. Thus $\omega = \alpha_1 \alpha_2$. Therefore the groups $\operatorname{im} i_{1\mathfrak{Q}}$ and $\operatorname{im} i_{2\mathfrak{Q}}$ generate the whole of $\mathfrak{Q}_F$. This completes the proof. $\square$

2.2 Bogomolov multipliers

2.2.1 Exterior squares

To measure the extent to which commutator relations in a given group G are consequences of the universal ones, we inspect the group obtained by factoring (G, G) by the images of universal commutator relations coming from homomorphisms from free groups to G. By Miller's theorem, it suffices to factor (G, G) by the normal subgroup generated by the relations (2.2). The resulting group is denoted by $G \wedge G$. In other words, $G \wedge G$

is the group generated by the symbols $x \wedge y$, where $x, y \in G$, subject to the following relations:

$$xy \wedge z = (x^y \wedge z^y)(y \wedge z), \quad x \wedge yz = (x \wedge z)(x^z \wedge y^z), \quad x \wedge x = 1 \tag{2.3}$$

with $x, y, z \in G$. The group $G \wedge G$ is said to be the *nonabelian exterior square* of G. There is a surjective homomorphism $\kappa\colon G \wedge G \to [G, G]$ defined by $x \wedge y \mapsto [x, y]$. The kernel $\mathrm{M}(G) = \ker \kappa$ is thus the group of commutator relations of G that are not consequences of the universal ones only.

***Example* 2.11.** *Take F to be a free group. Then $F \wedge F = (F, F)/\mathfrak{R}_F \cong [F, F]$ via the map $\kappa\colon (F, F) \to [F, F]$. Thus $\mathrm{M}(F) = 0$.*

***Example* 2.12.** *Take A to be an abelian group. Then $A \wedge A$ is also an abelian group, since $[x \wedge y, z \wedge w] = [x, z] \wedge [y, w] = 1$ by Lemma 2.10. Thus $A \wedge A$ is an abelian group generated by pairs $x \wedge y$ for $x, y \in A$ and its defining relations* (2.3) *can be rewritten as*

$$(x + y) \wedge z = (x \wedge z) + (y \wedge z), \quad x \wedge (y + z) = (x \wedge z) + (x \wedge y), \quad x \wedge x = 1$$

with $x, y, z \in A$. Therefore $A \wedge A$ is the ordinary exterior square of the abelian group A. Note that $A \wedge A$ may be much larger in size than A, and so $\mathrm{M}(A)$ can also be large.

The group $G \wedge G$ is a nonabelian version of the ordinary exterior square. It is a quotient of the nonabelian tensor square $G \otimes G$, which is the group obtained by factoring (G, G) by the relations (2.3) without the relations (x, x) for $x \in G$. In this context of generalizing a useful construction to nonabelian groups, both of these objects have been introduced by Brown and Loday [BL84, BL87] with applications in homotopy theory in mind.

2.2.2 Hopf's formula and Schur multipliers

The object $\mathrm{M}(G)$ has a homological interpretation. The basis for this is a combinatorial formula for $\mathrm{M}(G)$ coming from a free presentation of G.

Theorem 2.13 ([Mil52])**.** *Let G be a group given by a free presentation $G = F/R$. Then*

$$\mathrm{M}(G) \cong \frac{R \cap [F, F]}{[R, F]}.$$

Proof. Set $F_0 = F/[R, F]$ and $R_0 = R/[R, F]$. Then $G = F_0/R_0$ and R_0 is central in F_0. Define a homomorphism $\phi\colon (G, G) \to [F_0, F_0]$ by the rule $\phi(x, y) = [\tilde{x}, \tilde{y}]$, where $\tilde{x}, \tilde{y}$ are lifts of x, y in F_0. This induces a map $\colon G \wedge G \to [F_0, F_0]$ which restricts to an epimorphism $\bar{\phi}\colon \mathrm{M}(G) \to R_0 \cap [F_0, F_0]$. The kernel of $\bar{\phi}$ is precisely the image of $\mathrm{M}(F_0)$ in $\mathrm{M}(G)$. To see that this is trivial, take a $z = \prod_i (x_i[R, F] \wedge y_i[R, F]) \in \mathrm{M}(F_0)$ for some $x_i, y_i \in F$. Thus there exist $f_i \in F, r_i \in R$ so that $\prod_i [x_i, y_i] = \prod_i [r_i, f_i]^{\epsilon_i}$ for $\epsilon_i = \pm 1$. Since $\mathrm{M}(F) = 0$, the image of z in $\mathrm{M}(G)$ is the same as the image of $\prod_i (r_i \wedge f_i)^{\epsilon_i} \in \mathrm{M}(F)$ in $\mathrm{M}(G)$. The relation $[g, 1] = 1$ is universal, whence z is trivial in $\mathrm{M}(G)$. $\square$

Following the above proof, we see that the isomorphism given in the above theorem is induced by the canonical isomorphism $G \wedge G \to [F,F]/[R,F]$ given by $xR \wedge yR \mapsto [x,y][R,F]$.

The following result is well known.

Theorem 2.14 ([Hop42]). *Let G be a group given by a free presentation $G = F/R$. Then*

$$\mathrm{H}_2(G,\mathbb{Z}) \cong \frac{R \cap [F,F]}{[R,F]}.$$

Whence, given a group G, the group $\mathrm{M}(G)$ is naturally isomorphic to the group $\mathrm{H}_2(G,\mathbb{Z})$, also called the *Schur multiplier* of G. An explicit formula can be given for this isomorphism. Take $\omega = \prod_i (x_i, y_i) \in \mathfrak{R}_G$. The homology class in $\mathrm{H}_2(G,\mathbb{Z})$ corresponding to the image of ω in $\mathrm{M}(G)$ is the class of the 2-cycle

$$\rho(\omega) = \sum_i g(x_i,y_i) + \sum_i \left((\prod_{j=1}^{i} [x_j,y_j], [x_{i+1},y_{i+1}]) - (1,1) \right),$$

where $g(x,y) = (x,y) - (y,x) - (yx,(yx)^{-1}) + (xy,(yx)^{-1})$.

2.2.3 Removing redundancies

In light of Example 2.12, there may be many commutator relations of a given group G that are seemingly insignificant, but are nonetheless formally not consequences of the universal ones. The basic parts of these stem from commuting pairs in G. In order to remove these redundancies, we declare every commutator relation $(x,y) \in (G,G)$ to be *trivial*. Note that this leads to no effect in free groups, since elements there commute only for trivial reasons.

Let $\mathrm{M}_0(G) = \langle x \wedge y \mid x,y \in G,\, [x,y] = 1 \rangle$ be the group of trivial commutator relations among the relations that are not consequences of the universal ones. We now reduce the construction from before by modding out $\mathrm{M}_0(G)$. Define the *nonabelian curly exterior square* $G \curlywedge G$ of the group G by

$$G \curlywedge G = (G \wedge G)/\, \mathrm{M}_0(G).$$

There is an epimorphism $\kappa\colon G \curlywedge G \to [G,G]$ satisfying $x \curlywedge y \mapsto [x,y]$ with $x,y \in G$.

2.2.4 Bogomolov multipliers

Set

$$\mathrm{B}_0(G) = \ker \kappa = \mathrm{M}(G)/\, \mathrm{M}_0(G).$$

This group is called the *Bogomolov multiplier* of the group G. It is the group of nontrivial nonuniversal commutator relations, representing an obstruction for the commutator relations of G to follow from the universal commutator relations induced by (2.3) while considering the symbols that generate $\mathrm{M}_0(G)$ as redundant. The Bogomolov multiplier is the fundamental object of interest here. In the context of universal commutator relations, it was introduced recently by Moravec [Mor12] together with the nonabelian curly exterior square.

***Example* 2.15.** *Take A to be an abelian group. Then $\mathrm{M}_0(A) = \mathrm{M}(A)$, and therefore $A \curlywedge A = (A \wedge A)/\,\mathrm{M}(A) \cong [A, A] = 0$ and $\mathrm{B}_0(A) = 0$. Hence, all nonuniversal commutator relations of abelian groups are trivial.*

***Example* 2.16.** *Take G to be the quotient of a free group $F\langle x, y, z, w\rangle$ subject to the single relation $[x, y] = [z, w]$. Then $\mathrm{B}_0(G)$ is a cyclic group generated by the nontrivial relation $(x \wedge y)^{-1}(z \wedge w)$. We will formally show this later.*

The Bogomolov multiplier is an abelian group, since it is a quotient of the Schur multiplier. This fact can also be seen directly in terms of commutator relations. Let $\omega_1, \omega_2 \in \mathrm{B}_0(G)$ be two relations. Using Lemma 2.10, we have

$$[\omega_1, \omega_2] = \omega_1^{-1}\omega_1^{\omega_2} = \omega_1^{-1}\omega_1^{\kappa(\omega_2)} = \omega_1^{-1}\omega_1 = 1,$$

whence commutator relations commute in $\mathrm{B}_0(G)$.

Associating the exact sequence

$$0 \longrightarrow \mathrm{B}_0(G) \longrightarrow G \curlywedge G \xrightarrow{\kappa} [G, G] \longrightarrow 0$$

to a group G is a functor. If there is a homomorphism $\phi\colon G \to H$, then there exist homomorphisms $\mathrm{B}_0(\phi), \beta, \gamma$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{B}_0(G) \arrow[r] \arrow[d, "\mathrm{B}_0(\phi)"'] & G \curlywedge G \arrow[r] \arrow[d, "\beta"'] & {[G, G]} \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \mathrm{B}_0(H) \arrow[r] & H \curlywedge H \arrow[r] & {[H, H]} \arrow[r] & 0.
\end{tikzcd}$$

All homomorphisms stem from $\beta\colon G \curlywedge G \to H \curlywedge H$, defined by $x \curlywedge y \mapsto \phi(x) \curlywedge \phi(y)$.

2.2.5 Pairings

As with the ordinary tensor and exterior products, the definition of the curly exterior square can be given in terms of a universal property. Let L be a group. A function $\phi : G \times G \to L$ is called a B_0-*pairing* if for all $x, y, z \in G$, and for all $s, t \in G$ with $[s, t] = 1$,

$$\phi(xy, z) = \phi(x^y, z^y)\phi(y, z), \quad \phi(x, yz) = \phi(x, z)\phi(x^z, y^z), \quad \phi(s, t) = 1.$$

Clearly a B_0-pairing ϕ determines a unique homomorphism of groups $\phi^* : G \curlywedge G \to L$ such that $\phi^*(x \curlywedge y) = \phi(x, y)$ for all $x, y \in G$. The object $G \curlywedge G$ is universal with respect to having this property of producing a homomorphism from a B_0-pairing.

2.3 Motivation

Apparitions of the Bogomolov multiplier abound. In all cases, what matters most is whether or not the object itself is trivial. In this section, we present a context relating Bogomolov multipliers to Brauer groups developed by Bogomolov [Bog87]. It is based on this work that the object $\mathrm{B}_0(G)$ got its name and aroused interest for further investigation. Our exposition tightly follows [GS06].

2.3.1 Central simple algebras

Let k be a field. We assume throughout that all k-algebras are finite dimensional over k. An algebra A is said to be a *central simple algebra* if it is simple and $Z(A) = k$. The following is a classical result on scalar extensions of central simple algebras.

Theorem 2.17 ([GS06], Corollary 2.2.6)**.** *An algebra A is a central simple algebra if and only if there exists a positive integer n and a finite Galois extension K/k so that $A \otimes_k K \cong M_n(K)$.*

In the setting of the above theorem, we say that K is a *splitting field* for A. Thus, central simple algebras are the k-algebras that become isomorphic to a matrix algebra after a suitable extension of scalars.

2.3.2 Galois descent

Theorem 2.17 makes it possible to classify central simple algebras using methods of Galois theory. To achieve this, we work in the more general context of *vector spaces equipped with a tensor Φ of type (p, q)*. The tensor Φ is an element of $V^{\otimes p} \otimes_k (V^*)^{\otimes q} \cong \hom_k(V^{\otimes q}, V^{\otimes p})$ with $p, q \geq 0$. The case $p = 1, q = 2$ corresponds to k-bilinear maps $V \times V \to V$, i.e. specifying multiplication on V. Thus, let us consider pairs (V, Φ) of k-vector spaces equipped with a tensor of fixed type. An *isomorphism* between two such objects (V, Φ) and W, Ψ is given by an isomorphism $f\colon V \to W$ such that $f^{\otimes q} \otimes (f^*)^{-1}$ maps Φ to Ψ.

Now fix a finite Galois extension K/k with $G = \mathrm{Gal}(K/k)$. Given (V, Φ), set $V_K = V \otimes_k K$ and Φ_K the tensor induced on V_K by Φ. In this way we associate with (V, Φ) a K-object (V_K, Φ_K). We say that (V, Φ) and (W, Ψ) *become isomorphic* over K if there is an isomorphism between (V_K, Φ_K) and (W_K, Ψ_K). In this situation, (W, Ψ) is also called a *K-twisted form* of (V, Φ). The set of all K-twisted forms of (V, Φ) is denoted by $TF_K(V, \Phi)$.

One can classify isomorphism classes of twisted forms as follows. Given $\sigma \in \mathrm{Gal}(K/k)$, tensoring by V gives an isomorphism $\sigma\colon V_K \to V_K$. Each linear map $f\colon V_K \to W_K$ induces a map $\sigma(f)\colon V_K \to W_K$ by $\sigma(f) = \sigma \circ f \circ \sigma^{-1}$. Thus there is an action of $\mathrm{Gal}(K/k)$ on the group $\mathrm{Aut}_K(\Phi)$ of automorphisms of (V_K, Φ_K). Moreover, given an isomorphism $g\colon (V_K, \Phi_K) \to (W_K, \Psi_K)$, one gets a map $a\colon \mathrm{Gal}(K/k) \to \mathrm{Aut}_K(\Phi)$ by $a(\sigma) = g^{-1} \circ \sigma(g)$. It is easily verified that a satisfies the 1-cocycle condition $a(\sigma\tau) = a(\sigma) \cdot \sigma(a(\tau))$. Selecting another isomorphism $h\colon (V_K, \Phi_K) \to (W_K, \Psi_K)$ with a cocycle b, we have $a = c^{-1} b \sigma(c)$ for $c = h^{-1} \circ g$. Thus, one can associate to every

K-twisted form of (V,Φ) an element of the first cohomological group of $\mathrm{Gal}(K/k)$ with its action on $\mathrm{Aut}_K(\Phi)$. This map is a bijection and maps the class $M_n(K)$ into a cocycle that is cohomologous to a trivial one.

Theorem 2.18 ([GS06], Theorem 2.3.3)**.** *Let (V,Φ) be a k-object and K/k a Galois extension. There is a base-point preserving bijection*

$$TF_K(V,\Phi) \longleftrightarrow \mathrm{H}^1(\mathrm{Gal}(K/k),\mathrm{Aut}_K(\Phi)).$$

Proof. Main idea. To get the inverse, take a cocycle a and consider the vector space V equipped with a twisted action of $\mathrm{Gal}(K/k)$ by setting $\sigma(x) = a(\sigma)(\sigma(x))$. Denote this vector space by ${}_aV$. Now take $W = ({}_aV)^G$. Since $a(\sigma)(\sigma(\Phi_K)) = \Phi_K$, the tensor Φ_K comes from a tensor Ψ on W. Thus we associate to the cocycle a the object (W,Ψ). □

***Example* 2.19** (Hilbert's Theorem 90)**.** *Take V to be a vector space of dimension n over k and let Φ be the trivial tensor. Then* $\mathrm{Aut}_K(\Phi) = \mathrm{GL}_n(K)$*. As two vector spaces of the same dimension are isomorphic over K if and only if they are isomorphic over k, we get* $\mathrm{H}^1(\mathrm{Gal}(K/k),\mathrm{GL}_n(K)) = \{1\}$.

2.3.3 Brauer groups

Take a finite Galois extension K/k and let $G = \mathrm{Gal}(K/k)$. Set $\mathrm{CSA}_n(K)$ to be the set of isomorphism classes of central simple k-algebras that are split by K into $M_n(K)$. These are precisely the K-twisted forms of the matrix algebra $M_n(k)$, considered as an n^2-dimensional k-vector space equipped with a tensor of type $(2,1)$ satisfying the associativity condition. Note that $\mathrm{Aut}(M_n(k)) = \mathrm{PGL}_n(K)$. Theorem 2.18 thus gives the following.

Theorem 2.20. *There is a base-point preserving bijection*

$$\mathrm{CSA}_K(n) \longleftrightarrow \mathrm{H}^1(\mathrm{Gal}(K/k),\mathrm{PGL}_n(K)).$$

One can classify all central simple algebras split by K by means of a single cohomology set. To achieve this, consider the maps induced by tensoring $t_{nm}\colon\ \mathrm{PGL}_n(K)\times\mathrm{PGL}_m(K) \to \mathrm{PGL}_{nm}(K)$ for various n,m on the G-modules. These maps induce a natural product on cocycles, whence there is a product operation

$$\mathrm{H}^1(G,\mathrm{PGL}_n(K)) \times \mathrm{H}^1(G,\mathrm{PGL}_m(K)) \to \mathrm{H}^1(G,\mathrm{PGL}_{nm}(K)).$$

In terms of the isomorphism from Theorem 2.20, this corresponds to taking central simple algebras $A \in \mathrm{CSA}_n(K), B \in \mathrm{CSA}_m(K)$ and producing the algebra $A\otimes_k B \in \mathrm{CSA}_{nm}(K)$. Now, the natural maps $\bar{t}_{nm}\colon\ \mathrm{PGL}_m(K) \to \mathrm{PGL}_{nm}(K)$ defined by $\bar{t}_{nm}(A) = t_{nm}(I,A)$ induce maps

$$\lambda_{nm}\colon\ \mathrm{H}^1(G,\mathrm{PGL}_m(K)) \to \mathrm{H}^1(G,\mathrm{PGL}_{nm}(K))$$

on cohomology. In terms of central simple algebras, these correspond to taking $A \in \mathrm{CSA}_m(K)$ and producing $A \otimes_k M_n(k)$. If follows from Wedderburn's theorem that the maps λ_{nm} are injective. It is therefore natural to consider the direct limit

$$\mathrm{Br}(K/k) = \varinjlim_{n} \mathrm{CSA}_n(K),$$

which is called the *relative Brauer group* of K/k. Thus, elements of $\mathrm{Br}(K/k)$ are equivalence classes of central simple algebras split by K with respect to the relation $A \sim A'$ if $A \otimes_k M_m(k) \cong A' \otimes_k M_{m'}(k)$ for some m, m'. The direct limit of the sets $\mathrm{Br}(K/k)$ for all finite Galois extensions K/k is denoted by $\mathrm{Br}(k)$. This is the *Brauer group* of k. Based on the above discussion, both $\mathrm{Br}(K/k)$ and $\mathrm{Br}(k)$ are indeed groups, the inverse of a class of an algebra A is given by the class of the opposite algebra A^{op}.

Translating this construction to the level of cohomology, we take $\mathrm{PGL}_\infty(K) = \varinjlim_n \mathrm{PGL}_n(K)$ via the maps t_{mn} and consider the cohomology group

$$\mathrm{H}^1(G, \mathrm{PGL}_\infty(K)) \cong \mathrm{Br}(K/k).$$

Now take a limit when K varies over all finite Galois extensions of k. We obtain the Galois cohomology

$$\mathrm{H}^1(\mathrm{Gal}(k_s/k), \mathrm{PGL}_\infty(k_s)) \cong \mathrm{Br}(k)$$

of the profinite group $\mathrm{Gal}(k_s/k) = \varprojlim_K \mathrm{Gal}(K/k)$ with coefficients in $\mathrm{PGL}_\infty(k_s)$, where k_s is a separable closure of k. The cohomology here is taken to be continuous, meaning that the cocycles are continuous maps into the discrete module $\mathrm{PGL}_\infty(k_s)$. There is a more tractable description of the cohomology group $\mathrm{H}^1(\mathrm{Gal}(k_s/k), \mathrm{PGL}_\infty(k_s))$, resting on the following short cohomology sequence.

Lemma 2.21 ([GS06], Proposition 4.4.1)**.** *Let G be a group and*

$$1 \to A \to B \to C \to 1$$

a central extension of groups equipped with a G-action. Then there is an exact sequence of pointed sets

$$1 \to A^G \to B^G \to C^G \to \mathrm{H}^1(G, A) \to \mathrm{H}^1(G, B) \to \mathrm{H}^1(G, C) \to \mathrm{H}^2(G, A).$$

Proof. This is standard up to the final map in the sequence. To define $\partial\colon \mathrm{H}^1(G, C) \to \mathrm{H}^2(G, A)$, take a 1-cocycle $c\colon \sigma \mapsto c(\sigma)$ and lift each $c(\sigma)$ to an element $b(\sigma) \in B$. The cocycle relation implies that for all $\sigma, \tau \in G$, the element $b(\sigma)\sigma(b(\tau))b(\sigma\tau)^{-1}$ maps to 1 in C, hence comes from an element $a(\sigma, \tau) \in A$. Now take ∂ to map the class of c to the class of a. □

Applying the lemma to the exact sequence of G-groups

$$1 \to K^\times \to \mathrm{GL}_m(K) \to \mathrm{PGL}_m(K) \to 1$$

together with Hilbert's Theorem 90 yields an injection

$$\delta_m\colon\ \mathrm{H}^1(G, \mathrm{PGL}_m(K)) \to \mathrm{H}^2(G, K^\times).$$

These injections are compatible with the injections

$$\lambda_{nm}\colon \mathrm{H}^1(G, \mathrm{PGL}_m(K)) \to \mathrm{H}^1(G, \mathrm{PGL}_{nm}(K))$$

from above. Passing to the limit, there is thus an injection

$$\delta_\infty\colon \mathrm{H}^1(G, \mathrm{PGL}_\infty(K)) \to \mathrm{H}^2(G, K^\times).$$

This map transforms the multiplication defined above on the set $\mathrm{H}^1(G, \mathrm{PGL}_\infty(K))$ to multiplication on the cohomology group $\mathrm{H}^2(G, K^\times)$ and is also surjective.

Theorem 2.22 ([GS06], Theorem 4.4.5)**.** *There is a group isomorphism*

$$\delta_\infty\colon \mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{PGL}_\infty(K)) \to \mathrm{H}^2(\mathrm{Gal}(K/k), K^\times).$$

Thus there are natural isomorphisms

$$\mathrm{Br}(K/k) \cong \mathrm{H}^2(\mathrm{Gal}(K/k), K^\times) \quad \text{and} \quad \mathrm{Br}(k) \cong \mathrm{H}^2(\mathrm{Gal}(k_s/k), k_s^\times).$$

2.3.4 Cohomology of Laurent series fields

Set $G = \mathrm{Gal}(k_s/k)$ and suppose k is perfect. Consider the field of Laurent series $k_s((t))$. There is a valuation homomorphism $v\colon k_s((t)) \to \mathbb{Z}$ sending a Laurent series to the degree of the least nonzero term. Set $k((t))_{nr} = \varinjlim_K K((t))$ to be the maximal unramified extension of $k((t))$ and denote by U_{nr} the group of invertible power series contained in $k((t))_{nr}$. Thus there is a split exact sequence of G-modules

$$1 \to U_{nr} \to k((t))_{nr} \to \mathbb{Z} \to 0.$$

This induces a split exact sequence of cohomology groups

$$0 \to \mathrm{H}^2(G, U_{nr}) \to \mathrm{H}^2(G, k((t))_{nr}) \to \mathrm{H}^2(G, \mathbb{Z}) \to 0.$$

Since $\mathrm{H}^2(G, \mathbb{Z}) \cong \mathrm{H}^1(G, \mathbb{Q}/\mathbb{Z}) \cong \hom_{cont}(G, \mathbb{Q}/\mathbb{Z})$, the sequence may be rewritten as

$$0 \longrightarrow \mathrm{H}^2(G, U_{nr}) \longrightarrow \mathrm{H}^2(G, k((t))_{nr}) \xrightarrow{r_v} \hom_{cont}(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

The map r_v is called the *residue map* associated to v.

The kernel and the middle term of this map are naturally isomorphic to suitable Brauer groups.

Proposition 2.23 ([GS06], Proposition 6.3.1 and Proposition 6.3.4)**.** *The natural map $U_{nr} \to k_s^\times$ sending a power series to its constant term induces an isomorphism*

$$\mathrm{H}^2(G, U_{nr}) \cong \mathrm{H}^2(G, k_s^\times).$$

The inflation maps

$$\mathrm{H}^2(G, k((t))_{nr}^\times) \to \mathrm{H}^2(\mathrm{Gal}((k((t)))_s/k((t))), (k((t))_{nr})_s^\times)$$

are isomorphisms.

Thus we obtain the following decomposition of the Brauer group of the field of Laurent series.

Corollary 2.24 (Witt)**.** *For a perfect field k there is a split exact sequence*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(k((t))) \xrightarrow{r_v} \hom_{cont}(\mathrm{Gal}(k_s/k), \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

2.3.5 The rationality problem

Let $k(t_1, t_2, \ldots, t_n)$ be a purely transcendental extension of k. Suppose $k \subseteq K \subseteq k(t_1, t_2, \ldots, t_n)$ is a subfield such that $k(t_1, t_2, \ldots, t_n)/K$ is a finite extension. An old question asks as to whether or not the extension K/k is necessarily a purely transcendental extension. This is the *rationality problem.*

In the language of algebraic geometry, the problem may be rephrased as asking whether or not every unirational variety is rational. When $n = 1$, the answer is yes by a classical theorem of Lüroth [Šaf91] . In the case $n = 2$ counterexamples exist if k is not assumed to be algebraically closed, and the answer is again yes when k is algebraically closed of characteristic zero, see [GS06, Remarks 6.6.2]. In dimension 3, counterexamples were found by Artin and Mumford [AM72] over the complex ground field.

A natural way of producing extensions as in the rationality problem is as follows. One can identify a purely transcendental extension with the field of rational functions on a k-vector space V. If a finite group G acts on V, there is an induced action on $k(V)$. If the action is faithful, the extension $k(V)/k(V)^G$ is Galois with group G. Supposing that k is algebraically closed and of characteristic 0, counterexamples to the rationality problem did not rule out the possibility that $k(V)^G/k$ might always be purely transcendental. This weaker version of the rationality problem is known as *Noether's problem* [Noe13]. Saltman [Sal84] showed that the answer to this question is in general negative. His approach was inspired by that of Artin and Mumford, and developed further in works of Bogomolov [Bog87]. We will take a closer look at this construction and show how it relates to commutator relations.

The starting point is considering the following invariant. Let K be a Galois extension of k and consider a discrete valuation ring A with fraction field K. The completion of K is isomorphic to a Laurent series field $\kappa((t))$ with κ being the residue field of A. The Brauer group $\mathrm{Br}(\kappa((t)))$ may be understood in terms of the residue map

$$r_A \colon \mathrm{Br}(\kappa((t))) \to \hom_{cont}(\mathrm{Gal}(\kappa_s/\kappa), \mathbb{Q}/\mathbb{Z}).$$

By precomposing with the natural map $\mathrm{Br}(K) \to \mathrm{Br}(\kappa((t)))$, we get a composite map

$$r_A \colon \mathrm{Br}(K) \to \hom_{cont}(\mathrm{Gal}(\kappa_s/\kappa), \mathbb{Q}/\mathbb{Z}).$$

We may therefore understand $\mathrm{Br}(K)$ in terms of the maps r_A. The loss of information by doing so is contained in the intersection $\cap_A \ker(r_A)$ over all discrete valuation rings A. This intersection is called the *unramified Brauer group* of K and denoted by $\mathrm{Br}_{nr}(K)$.

The crucial property of the unramified Brauer group for the rationality problem is that it is invariant under purely transcendental extensions.

Proposition 2.25 ([GS06], Proposition 6.6.6)**.** *Let $k(t)/k$ be a purely transcendental extension. Then the natural map $\mathrm{Br}_{nr}(k) \to \mathrm{Br}_{nr}(k(t))$ is an isomorphism.*

In particular, if k is algebraically closed, then $\mathrm{Br}_{nr}(k(t_1, t_2, \ldots, t_n)) = 0$. The strategy now is to find some group G with an action on a vector space V such that $\mathrm{Br}_{nr}(k(V)^G)$ is nontrivial. This will imply that $k(V)^G/k$ is not purely transcendental. We will need to do better than taking abelian groups.

Theorem 2.26 ([GS06], Theorem 6.6.8)**.** *Let A be a finite abelian group acting faithfully on a k-vector space V. Then the extension $k(V)^A/k$ is purely transcendental.*

Proof. The A-module V is semisimple, so it decomposes into a sum of 1-dimensional A-modules. Pick a vector v_i in each of these components and let $X = \langle v_i \rangle \leq k(V)^\times$ be the subgroup generated by these vectors. Now consider the map from X to A^* associating v_i to the character that corresponds to the A-action on the component to which v_i belongs. Let Y be the kernel of this map, so that $Y \subseteq k(V)^A$. Since $|A| = |k(V) : k(V)^A| \leq |k(V) : k(Y)| \leq |A|$, it follows that $k(Y) = k(V)^A$. The proof is now complete, since Y is free abelian. □

Therefore elements of $\mathrm{Br}_{nr}(k(V)^G)$ vanish after restricting to $\mathrm{Br}_{nr}(k(V)^A)$ for any abelian group $A \leq G$. Bogomolov showed that this feature characterizes the unramified Brauer group.

Theorem 2.27 ([Bog87])**.** *Let G be a finite group acting faithfully on a k-vector space V. Then*

$$\mathrm{Br}_{nr}(k(V)^G) = \bigcap_{A \in \mathcal{A}} \ker\left(\mathrm{Br}(k(V)^G) \to \mathrm{Br}(k(V)^A)\right),$$

where $\mathcal{A}$ is the set of all abelian subgroups of G.

Using the above description, Bogomolov was moreover able to give a purely group-theoretical characterization of $\mathrm{Br}_{nr}(k(V)^G)$.

Theorem 2.28 ([Bog87])**.** *Let G be a finite group acting faithfully on a k-vector space V. Then*

$$\mathrm{Br}_{nr}(k(V)^G) \cong \bigcap_{A \in \mathcal{A}} \ker\left(\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(A, \mathbb{Q}/\mathbb{Z})\right),$$

where $\mathcal{A}$ is the set of all abelian subgroups of G.

Main idea. There is an isomorphism $\mathrm{Br}(k(V)/k(V)^G) \cong \mathrm{H}^2(G, k(V)^\times)$. The embedding of $\mathrm{Br}(k(V)/k(V)^G)$ into $\mathrm{Br}(k(V)^G)$ surjects onto the unramified elements. Therefore it suffices to consider the claim for the case when Brauer groups are replaced by ordinary cohomology with coefficients in $k(V)^\times$. To change this module to $k^\times$, write the long exact sequence corresponding to $0 \to k^\times \to k(V)^\times \to \mathrm{Div}(\mathbb{A}^n_k) \to 0$, decompose $\mathrm{Div}(\mathbb{A}^n_k)$ and use Shapiro's lemma. □

Observe that $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$ is the Schur multiplier of G. The above description can be dualized into a homological version, finally giving the following.

Theorem 2.29 ([Mor12]). *Let G be a finite group acting faithfully on a k-vector space V. Then*

$$\mathrm{Br}_{nr}(k(V)^G)^* \cong \mathrm{B}_0(G).$$

Proof. We have

$$\mathrm{Br}_{nr}(k(V)^G)^* \cong \left(\bigcap_{A \in \mathcal{A}} \ker \left(\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(A, \mathbb{Q}/\mathbb{Z}) \right) \right)^* .$$

This object is a quotient of $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})^*$ by the subgroup generated by the images of homomorphisms $\mathrm{H}^2(A, \mathbb{Q}/\mathbb{Z})^* \to \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})^*$ for all $A \in \mathcal{A}$. Dualizing to homology, we obtain

$$\mathrm{Br}_{nr}(k(V)^G)^* \cong \mathrm{H}_2(G, \mathbb{Z}) / \left\langle \mathrm{im}\left(\mathrm{H}_2(A, \mathbb{Z}) \to \mathrm{H}_2(G, \mathbb{Z})\right) \mid A \in \mathcal{A} \right\rangle .$$

Identifying $\mathrm{H}_2(G, \mathbb{Z}) \cong \mathrm{M}(G)$, the latter object is the same as the quotient of $\mathrm{M}(G)$ by the subgroup $\langle \mathrm{im}\,(\mathrm{M}(A) \to \mathrm{M}(G)) \mid A \in \mathcal{A} \rangle$. Finally, note that

$$\langle \mathrm{im}\,(\mathrm{M}(A) \to \mathrm{M}(G)) \mid A \in \mathcal{A} \rangle \cong \langle \mathrm{im}\,(\mathrm{M}(\langle x, y \rangle) \to \mathrm{M}(G)) \mid x, y \in G,\ [x, y] = 1 \rangle,$$

and so

$$\mathrm{Br}_{nr}(k(V)^G)^* \cong \mathrm{M}(G) / \mathrm{M}_0(G) \cong \mathrm{B}_0(G).$$ □

Thus, there are counterexamples for the rationality problem coming from groups that possess nontrivial nonuniversal commutator relations. The group given in Example 3.20 is one of the first examples of Saltman [Sal84] in this context.

3

Basic properties and examples

Some basic properties of the Bogomolov multiplier are explored. We give a Hopf-type formula, show that it is invariant under isoclinism, and investigate behavior with respect to taking quotients and subgroups. Many examples illustrating different techniques in dealing with concrete classes of groups are presented.

This chapter is based on [FAJ, Jez14, Mor12, Mor14].

3.1 Basic properties

3.1.1 A Hopf-type formula

Let G be given by a free presentation F/R. We will derive a description of $\mathrm{B}_0(G)$ in terms of the isomorphism from Theorem 2.13. We have that $\mathrm{M}(G) \cong (R \cap [F,F])/[R,F]$. Consider the isomorphism $G \wedge G \to [F,F]/[R,F]$ given by $xR \wedge yR \mapsto [x,y][R,F]$. Under this map, $\mathrm{M}_0(G)$ can be identified with the subgroup of $F/[F,R]$ generated by all the commutators in $F/[F,R]$ that are relations in G. In other words, we have that $\mathrm{M}_0(G) \cong \langle \mathrm{K}(F/[R,F]) \cap R/[R,F] \rangle = \langle \mathrm{K}(F) \cap R \rangle [R,F]/[R,F] = \langle \mathrm{K}(F) \cap R \rangle/[R,F]$. Thus we have the following Hopf-type formula for $\mathrm{B}_0(G)$.

Proposition 3.1 ([Mor12])**.** *Let G be a group given by a free presentation $G = F/R$. Then*

$$\mathrm{B}_0(G) \cong \frac{[F,F] \cap R}{\langle \mathrm{K}(F) \cap R \rangle}.$$

In some cases, this formula enables explicit calculations of $\mathrm{B}_0(G)$, given a free presentation of G.

***Example* 3.2.** *A word w in a free group F is said to be a* commutator word *if $w = [u,v]$ for some $u, v \in F$. Let $\mathfrak{V}$ be a variety of groups defined by a commutator word w. If G is a $\mathfrak{V}$-relatively free group, then* $\mathrm{B}_0(G) = 0$*. To see this, take a presentation $F/\mathfrak{V}(F)$ of the group G as a quotient of a free group F by the verbal subgroup $\mathfrak{V}(F)$ of F. Note that $\mathfrak{V}(F) \leq [F,F]$ and $\langle \mathrm{K}(F) \cap \mathfrak{V}(F) \rangle = \mathfrak{V}(F)$. Our claim now follows from Proposition 3.1.*

Some more involved examples will be given in Section 3.2.

3.1.2 Isoclinism

The Bogomolov multiplier of an abelian group is trivial. More generally, parts of a group where elements commute will not contribute to producing nontrivial commutator relations. The most general context to place such a situation in is the notion of isoclinism introduced by P. Hall [Hal40] with the intention of trying to bring order into the complicated world of finite p-groups.

Two groups G and H are *isoclinic* if there exists a pair of isomorphisms $\alpha\colon G/Z(G) \to H/Z(H)$ and $\beta\colon [G,G] \to [H,H]$ with the property that whenever $\alpha(x_1 Z(G)) = x_2 Z(H)$ and $\alpha(y_1 Z(G)) = y_2 Z(H)$, then $\beta([x_1, y_1]) = [x_2, y_2]$ for $x_1, y_1 \in G$. Isoclinism is an equivalence relation, denoted by the symbol $\simeq$, and the equivalence classes are called *families*. Hall proved that each family contains *stem groups*, that is, groups G satisfying $Z(G) \leq [G,G]$. Stem groups in a given family have the same order, which is the minimal order of all groups in the family. When the stem groups are of order p^r for some r, we call r the *rank* of the family.

Bogomolov multipliers are invariant with respect to isoclinism.

Theorem 3.3 ([Mor14])**.** *Let G and H be isoclinic groups. Then* $\mathrm{B}_0(G) \cong \mathrm{B}_0(H)$.

Proof. There exist isomorphisms $\alpha : G/Z(G) \to H/Z(H)$ and $\beta : [G,G] \to [H,H]$ satisfying the compatibility conditions. Define a map $\phi : G \times G \to H \curlywedge H$ by $\phi(x_1, y_1) = x_2 \curlywedge y_2$, where x_i, y_i are as above. It is readily verified that this map is well defined.

Suppose that $x_1, y_1 \in G$ commute, and let $x_2, y_2 \in H$ be as above. By definition, $[x_2, y_2] = \beta([x_1, y_1]) = 1$, hence $x_2 \curlywedge y_2 = 1$. This, and the relations of $H \curlywedge H$, ensure that ϕ is a B_0-pairing. Thus ϕ induces a homomorphism $\gamma\colon G \curlywedge G \to H \curlywedge H$ such that $\gamma(x_1 \curlywedge y_1) = x_2 \curlywedge y_2$ for all $x_1, y_1 \in G$. By symmetry there exists a homomorphism $\delta\colon H \curlywedge H \to G \curlywedge G$ defined via α^{-1}. It is straightforward to see that δ is the inverse of γ, hence γ is an isomorphism.

Let $\kappa_1\colon G \curlywedge G \to [G,G]$ and $\kappa_2\colon H \curlywedge H \to [H,H]$ be the commutator maps. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{B}_0(G) & \longrightarrow & G \curlywedge G & \xrightarrow{\kappa_1} & [G,G] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\tilde{\gamma}} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & \mathrm{B}_0(H) & \longrightarrow & H \curlywedge H & \xrightarrow{\kappa_2} & [H,H] & \longrightarrow & 0
\end{array}$$

Here $\tilde{\gamma}$ is the restriction of γ to $\mathrm{B}_0(G)$. Since β and γ are isomorphisms, so is $\tilde{\gamma}$. This concludes the proof. $\square$

***Example* 3.4.** *Take G to be any group and A an abelian group. Then $G \simeq G \times A$, and so* $\mathrm{B}_0(G) \cong \mathrm{B}_0(G \times A)$.

3.1.3 Multiplicativity

Theorem 3.5 ([Kan14])**.** *Let G and H be groups. Then* $\mathrm{B}_0(G \times H) \cong \mathrm{B}_0(G) \times \mathrm{B}_0(H)$.

Proof. Since the Bogomolov multiplier is an abelian group, there is a natural homomorphism $\alpha\colon \mathrm{B}_0(G)\times \mathrm{B}_0(H) \to \mathrm{B}_0(G\times H)$ induced by inclusions. This is an epimorphism, since every element $(x,y)\curlywedge(z,w) \in (G\times H)\curlywedge(G\times H)$ can be expanded in terms of elements belonging to the subgroups $G\curlywedge G$ and $H\curlywedge H$ by using the universal commutator relations together with the fact that G and H commute in $G\times H$. The inverse to the map α can be constructed by specifying a B_0-pairing $\phi\colon (G\times H)\times(G\times H) \to (G\curlywedge G)\times(H\curlywedge H)$ by the rule $((x,y),(z,w)) \mapsto (x\curlywedge z, y\curlywedge w)$. This pairing induces a homomorphism $\beta\colon \mathrm{B}_0(G\times H) \to \mathrm{B}_0(G)\times \mathrm{B}_0(H)$, giving an inverse to α. □

3.1.4 Quotients and a 5-term exact sequence

Let G be a group and N a normal subgroup of G. There is a natural epimorphism $G \to G/N$. It is not straightforward to deduce what $\mathrm{B}_0(G/N)$ is from knowing $\mathrm{B}_0(G)$, since the induced homomorphism $\mathrm{B}_0(G)\to \mathrm{B}_0(G/N)$ may not even be surjective. The defect of this is measured with the help of a five term exact sequence associated to the short exact sequence $1\to N\to G\to G/N\to 1$ of groups. This is an analogue of the well known five term homology sequence, cf. [Bro82, p. 46].

Theorem 3.6 ([Mor12])**.** *Let G be a group and N a normal subgroup of G. Then there is an exact sequence*

$$\mathrm{B}_0(G) \longrightarrow \mathrm{B}_0(G/N) \longrightarrow \frac{N}{\langle \mathrm{K}(G)\cap N\rangle} \longrightarrow G^{\mathrm{ab}} \longrightarrow (G/N)^{\mathrm{ab}} \longrightarrow 0.$$

Proof. Let G have a free presentation $G = F/R$, and let S/R be the corresponding free presentation of N. Then $\mathrm{B}_0(G) \cong ([F,F]\cap R)/\langle \mathrm{K}(F)\cap R\rangle$ and $\mathrm{B}_0(G/N) \cong ([F,F]\cap S)/\langle \mathrm{K}(F)\cap S\rangle$. The epimorphism $\rho\colon G\to G/N$ induces a homomorphism $\rho^*\colon \mathrm{B}_0(G)\to \mathrm{B}_0(G/N)$. Observe that

$$\ker\rho^* = \frac{R\cap\langle \mathrm{K}(F)\cap S\rangle}{\langle \mathrm{K}(F)\cap R\rangle}$$

and

$$\operatorname{im}\rho^* = \frac{[F,F]\cap\langle \mathrm{K}(F)\cap S\rangle R}{\langle \mathrm{K}(F)\cap S\rangle}.$$

Since $N/\langle \mathrm{K}(G)\cap N\rangle \cong S/\langle \mathrm{K}(F)\cap S\rangle R$, there is a natural map $\sigma : \mathrm{B}_0(G/N) \to N/\langle \mathrm{K}(G)\cap N\rangle$. We have that $\ker\sigma = \operatorname{im}\rho^*$ and

$$\operatorname{im}\sigma = \frac{[F,F]\cap S)R}{\langle \mathrm{K}(F)\cap S\rangle R} = \frac{[F,F]R\cap S}{\langle \mathrm{K}(F)\cap S\rangle R} = \frac{[G,G]\cap N}{\langle \mathrm{K}(G)\cap N\rangle}.$$

Furthermore, there is a natural map $\pi\colon N/\langle \mathrm{K}(G)\cap N\rangle \to G^{\mathrm{ab}}$ whose kernel is equal to $\operatorname{im}\sigma$, and $\operatorname{im}\pi = N[G,G]/[G,G]$. Finally, there is a surjective homomorphism $G^{\mathrm{ab}}\to (G/N)^{\mathrm{ab}}$ whose kernel is equal to $\operatorname{im}\pi$. Our assertion now readily follows. □

Let us record a useful immediate corollary.

Corollary 3.7. *Let G be a group and N a normal subgroup of G that is generated by commutators. Then* $\mathrm{B}_0(G)$ *naturally surjects onto* $\mathrm{B}_0(G/N)$*.*

***Example* 3.8.** *We are now able to deal with Example 2.16. Take G to be the quotient of a free group $F\langle x, y, z, w\rangle$ subject to the single relation $[x, y] = [z, w]$. Set $N = \langle\{([x, y]^{-1}[z, w])^t \mid t \in F\}\rangle$, so that $G = F/N$. We claim that $\mathrm{B}_0(G)$ is a cyclic group generated by the nontrivial relation $(x \curlywedge y)^{-1}(z \curlywedge w)$ of infinite order. To see this, use the 5-term exact sequence. Since $\mathrm{B}_0(F) = 0$, we obtain an exact sequence*

$$0 \longrightarrow \mathrm{B}_0(G) \longrightarrow \frac{N}{\langle \mathrm{K}(F) \cap N\rangle} \longrightarrow F^{\mathrm{ab}} \longrightarrow G^{\mathrm{ab}} \longrightarrow 0.$$

As $F^{\mathrm{ab}} \cong G^{\mathrm{ab}}$, it follows that $\mathrm{B}_0(G) \cong N/\langle \mathrm{K}(F) \cap N\rangle$, which is in turn isomorphic to $\langle [x, y]^{-1}[z, w]\rangle \cong \mathbb{Z}$.

More can be said about curly exterior squares of quotients.

Proposition 3.9 ([Mor12]). *Let G be a group and N a normal subgroup of G. Then $G/N \curlywedge G/N \cong (G \curlywedge G)/J_N$, where $J_N = \langle x \curlywedge y \mid x, y \in G, [x, y] \in N\rangle$.*

Proof. There is a B_0-pairing $G/N \times G/N \to (G \curlywedge G)/J_n$ given by $(xN, yN) \mapsto (x \curlywedge y)J_N$. It induces a homomorphism $\alpha\colon G/N \curlywedge G/N \to (G \curlywedge G)/J_N$. On the other hand, there is also a B_0-pairing $G \times G \to G/N \curlywedge G/N$ that induces a homomorphism $G \curlywedge G \to G/N \curlywedge G/N$. Under this homomorphism J_N gets mapped to 1. The induced homomorphism provides an inverse to α. □

3.1.5 Subgroups and Sylow subgroups

Let G be a group and H a subgroup of G. There is a natural embedding $H \to G$. As with quotients, the induced homomorphism $\mathrm{B}_0(H) \to \mathrm{B}_0(G)$ may not be injective.

***Example* 3.10.** *Let G be a finite p-group with $\mathrm{B}_0(G) \neq 0$. We will see in Section 3.2 that there are plenty of examples of such groups. Take an embedding of G into a sufficiently large group $\mathrm{UT}_n(\mathbb{F}_p)$ of unitriangular matrices over a field of order p. We will show in Section 3.2 that $\mathrm{B}_0(\mathrm{UT}_n(\mathbb{F}_p)) = 0$. Hence the induced map $\mathrm{B}_0(G) \to \mathrm{B}_0(\mathrm{UT}_n(\mathbb{F}_p))$ is trivial.*

Bogomolov multipliers of subgroups are more difficult to handle than quotients. We do not know of a result analogous to the one of the 5-term exact sequence that would apply in the subgroup situation. In the case of finite groups, there exists, however, a general reduction to finite p-groups via Sylow's theory. This result is based on the homological interpretation of the Bogomolov multiplier, cf. [BT82, Proposition 6.9], and is to be compared with [BMP04, Lemma 2.6].

Theorem 3.11. *Let G be a finite group and P a Sylow p-subgroup of G. Then the image of the canonical induced map $\mathrm{B}_0(P) \to \mathrm{B}_0(G)$ is the Sylow p-subgroup $\mathrm{B}_0(G)_p$ of $\mathrm{B}_0(G)$. Moreover, $\mathrm{B}_0(G)_p$ is isomorphic to a direct summand of $\mathrm{B}_0(P)$.*

Proof. Set $m = |G : P|$. Let $i\colon P \to G$ be the natural inclusion. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{M}(G) \arrow[r, "j"] & \mathrm{M}(P) \arrow[r, "i_*"] & \mathrm{M}(G) \\
\mathrm{H}_2(G,\mathbb{Z}) \arrow[u, "\iota"] \arrow[r, "\mathrm{Cor}"] & \mathrm{H}_2(P,\mathbb{Z}) \arrow[u, "\iota"] \arrow[r, "\mathrm{Res}"] & \mathrm{H}_2(G,\mathbb{Z}) \arrow[u, "\iota"] \\
\mathrm{H}^2(G,\mathbb{C}^\times)^* \arrow[u, "\nu"] \arrow[r, "\mathrm{Cor}^*"] & \mathrm{H}^2(P,\mathbb{C}^\times)^* \arrow[u, "\nu"] \arrow[r, "\mathrm{Res}^*"] & \mathrm{H}^2(G,\mathbb{C}^\times)^* \arrow[u, "\nu"]
\end{tikzcd}$$

with ι and ν isomorphisms and $j = \iota^{-1}\,\mathrm{Cor}\,\iota$. The isomorphism ν comes from the Universal Coefficient Theorem. The composition of corestriction with restriction is the same as multiplication by m on the group $\mathrm{M}(G)$. As $\mathrm{M}(P)$ is a p-group and multiplication by m induces an automorphism on $\mathrm{M}(G)_p$, it follows that $\operatorname{im} i_* = \mathrm{M}(G)_p$. Moreover, $\mathrm{M}(P)$ is a direct sum of $\ker i_*$ and $\mathrm{M}(G)_p$. Now, a trivial relation $x \wedge y \in \mathrm{M}_0(G)$ corresponds to the dual of a symmetric cocycle in $\mathrm{H}^2(G,\mathbb{C}^\times)^*$, cf. 4.8. This property is conserved under the dual of the cohomological corestriction map. Therefore the diagram above factors through $\mathrm{M}_0(G)$ and $\mathrm{M}_0(P)$. The result thus follows from the one for the Schur multiplier. $\square$

***Example* 3.12.** *Take G to be a group with all Sylow subgroups abelian, for example $G = \mathrm{PSL}_2(5) \cong A_5$. Then $\mathrm{B}_0(G) = 0$. It may be that $\mathrm{M}(G)$ is not trivial. This happens in the case of A_5, where there is a trivial commutator relation $[(1,2)(3,4),(1,3)(2,4)] = 1$ that is not universal.*

3.2 Examples

3.2.1 Abelian-by-cyclic groups

Theorem 3.13. *Let G be a finite group with an abelian normal subgroup A such that G/A is cyclic. Then $\mathrm{B}_0(G) = 0$.*

The same theorem has been proved in [Bog87] using an alternative description of the Bogomolov multiplier.

Proof. By Theorem 3.11, it suffices to consider finite p-groups. Let $G = \langle A, g\rangle$ for some $x \in G$. Take $x \curlywedge y \in G \curlywedge G$ and write $x = ag^i, y = g^j a'$ for some integers i, j and $a, a' \in A$. As G is nilpotent, we have

$$a \curlywedge g^j a' = (a \curlywedge g^j)^{a'} = a \curlywedge g^j [g^j, a'] = (a \curlywedge g^j)^{[g^j,a']} = \cdots = a \curlywedge g^j,$$

and then

$$x \curlywedge y = (a \curlywedge g^j a')^{g^i} (g^i \curlywedge g^j a') = (a \curlywedge g^j)^{g^i} (g^i \curlywedge a').$$

Now, writing $a'' = a^{g^i}$, we have $x \curlywedge y = (a'' \curlywedge g^j)(g^i \curlywedge a')$. Expanding the powers of g, this may further be rewritten into a product of terms $g \curlywedge \tilde{a}$ for some $\tilde{a} \in A$. Using nilpotency, such a product can be collected into a single term of the same form. Thus, any element $\omega \in \mathrm{B}_0(G)$ can be written as $\omega = g \curlywedge \tilde{a}$ for some $\tilde{a} \in A$. Therefore ω is trivial. $\square$

***Example* 3.14.** *Let D be a dihedral group. Then $\mathrm{B}_0(D) = 0$. In the case when $|D| = 2^n$, the group D is isoclinic to the semi-dihedral group and the generalized quaternion group, so both of these also have trivial Bogomolov multipliers.*

3.2.2 Symmetric groups

Theorem 3.15 ([Noe13])**.** *Let S_n be the symmetric group. Then $\mathrm{B}_0(S_n) = 0$.*

Proof. Let $[n] = \{1, 2, \dots, \}$ be the set on which S_n naturally acts. Take V to be the $\mathbb{Q}$-vector space with basis $[n]$. The rational field $\mathbb{Q}(V)$ is isomorphic to the field of rational functions in n variables. Thus the fixed field $\mathbb{Q}(V)^{S_n}$ is the field of symmetric rational functions. This is a purely transcendental extension of $\mathbb{Q}$ whose basis consists of elementary symmetric polynomials. Thus $\mathrm{B}_0(S_n) \cong \mathrm{Br}_{nr}(\mathbb{Q}(V)^{S_n}) = 0$. $\square$

3.2.3 Finite simple groups

Theorem 3.16 ([Kun10])**.** *Let G be a finite quasi-simple group. Then $\mathrm{B}_0(G) = 0$.*

Main idea. Set $\tilde{G}$ to be the universal cover of G. It suffices to prove that every element of $Z(\tilde{G})$ is a commutator. This may be done by applying the results of Blau [Bla94] who classified all central elements having a fixed point in the natural action on the set of conjugacy classes of $\tilde{G}$. Such elements evidently admit the stated representation as a commutator. $\square$

3.2.4 Burnside groups

Theorem 3.17 ([Mor12])**.** *Suppose $m > 1$ and let $n > 2^{48}$ be odd. Let $B(m, n)$ be the free Burnside group of rank m and exponent n. Then $\mathrm{B}_0(B(m, n))$ is free abelian of countable rank.*

Proof. Ivanov [Iva94] showed that all centralizers of nontrivial elements of $B(m, n)$ are cyclic. From here it follows that $\mathrm{B}_0(B(m, n)) \cong \mathrm{H}_2(B(m, n), \mathbb{Z})$. The latter group is free abelian of countable rank, cf. [Ols91, Corollary 31.2]. $\square$

3.2.5 Unitriangular groups

The Schur multiplier of the general linear group $\mathrm{GL}_n(\mathbb{F}_p)$ is trivial whenever $(n, p) \notin \{(3, 2), (4, 2)\}$, and isomorphic to $\mathbb{Z}/2\mathbb{Z}$ otherwise [ERJ08]. On the other hand, the structure of the multipliers of the Sylow subgroups of $\mathrm{GL}_n(\mathbb{F}_p)$ is a lot more abundant. The Sylow p-subgroup corresponds to the unitriangular group $\mathrm{UT}_n(\mathbb{F}_p)$, and the q-subgroups for $q \neq p$ are given as wreath products of the unitriangular group and cyclic groups [AHN05]. Using Blackburn's results [Bla72], calculating the multipliers of the latter groups is routine once the multiplier of the unitriangular group is determined.

It turns out that an explicit formula can be given for the Schur multiplier of the unitriangular group over $\mathbb{Z}/m\mathbb{Z}$ for all integers m. We rely here on work done previously concerning the mod-p-multiplier of unitriangular groups [Eve72, BD01].

Theorem 3.18. *The Schur multiplier of* $\mathrm{UT}_n(\mathbb{Z}/m\mathbb{Z})$ *is isomorphic to*

$$C_m{}^{\binom{n}{2}-1} \text{ for odd } m, \qquad C_2{}^{n-3} \oplus C_{\frac{m}{2}}{}^{n-2} \oplus C_m{}^{\binom{n-1}{2}} \text{ for even } m.$$

Proof. This isomorphism stems from Hopf's formula. To apply it, we take a presentation of the unitriangular group as follows (see [BD01]). The set of elementary matrices $I + E_{i,i+1}$ generates the unitriangular group. Set $\mathcal{S} = \{s_i \mid 1 \le i \le n-1\}$ and let $\mathcal{R}$ be the set of relators in Table 3.1. Furthermore, relators in the final row of this table are unnecessary for odd m.

Table 3.1: Relators in the presentation of $\mathrm{UT}_n(\mathbb{Z}/m\mathbb{Z})$.

s_i^m	$1 \le i \le n-1$
$[s_i, s_j]$	$1 \le i < j-1 \le n-2$
$[s_i, s_{i+1}, s_i]$	$1 \le i \le n-2$
$[s_i, s_{i+1}, s_{i+1}]$	$1 \le i \le n-2$
$[[s_i, s_{i+1}], [s_{i+1}, s_{i+2}]]$	$1 \le i \le n-3$

Now, let F be the free group on $\mathcal{S}$ and R the normal subgroup of F generated by $\mathcal{R}$, so that F/R is a free presentation of the group $\mathrm{UT}_n(\mathbb{Z}/m\mathbb{Z})$. Its Schur multiplier is then given by the formula $(R \cap [F,F])/[R,F]$. It is a matter of a simple calculation to restrict the orders of some special elements of this group, gathered in Table 3.2.

Table 3.2: Generators of $\mathrm{M}(\mathrm{UT}_n(\mathbb{Z}/m\mathbb{Z}))$ and their orders.

$[s_i, s_j]$	m
$[s_i, s_{i+1}, s_i]$	m
$[s_i, s_{i+1}, s_i s_{i+1}^{-1}]$	$\gcd(m, \binom{m}{2})$
$[[s_i, s_{i+1}], [s_{i+1}, s_{i+2}]]$	$\gcd(2, m)$

From here on, it is easy to deduce that the relators of Table 3.2 generate the group $(R \cap [F,F])/[R,F]$. It is, however, more challenging to prove that they are also independent. The calculations are collected in [Jez14]. □

Corollary 3.19. $\mathrm{B}_0(\mathrm{UT}_n(\mathbb{Z}/m\mathbb{Z})) = 0$.

Proof. Immediate, since the generators of the Schur multiplier are commutators. □

A more direct approach akin to the one used before by computing in the curly exterior square can also be used to deduce that $\mathrm{B}_0(\mathrm{UT}_n(\mathbb{Z}/m\mathbb{Z})) = 0$. This has been done in [Mic13]. By developing the theory of Bogomolov multipliers further, we will see that there is in fact a very short argument to show this.

3.2.6 Small p-groups

It should be clear by now that it is more demanding to provide nontrivial examples than trivial. Based on Theorem 3.11, we focus on finite p-groups to provide such examples. Special cases of these which are easiest to handle are groups of tiny nilpotency classes and groups of small orders.

Studying Bogomolov multipliers of p-groups of nilpotency class 2 and exponent p (assume $p > 2$) can be translated into a problem in linear algebra over finite fields as follows (cf. [Bog87]).

Suppose that such a group G is of Frattini rank d, so that $G/[G,G] \cong \mathbb{F}_p^d$. The structure of G is then completely determined by the set of relations between commutators. These may be thought of as elements of the vector space $\mathbb{F}_p^d \wedge \mathbb{F}_p^d$ via the correspondence $[x, y] \equiv x \wedge y$. Selecting a basis $\{z_i \mid 1 \le i \le d\}$ of $\mathbb{F}_p^d$ gives a basis $\{z_i \wedge z_j \mid 1 \le i < j \le d\}$ of $\mathbb{F}_p^d \wedge \mathbb{F}_p^d$. The set of all relations between commutators in G forms a certain linear subspace R in $\mathbb{F}_p^d \wedge \mathbb{F}_p^d$. In this sense, commuting pairs in G correspond to decomposable elements of R, i.e. elements of the form $x \wedge y$ for some $x, y \in \mathbb{F}_p$. Such elements of the ambient vector space $\mathbb{F}_p^d \wedge \mathbb{F}_p^d$ are precisely the points on the algebraic variety $\mathfrak{P}$ determined by the Plücker relations. In the case $d = 4$, these form a single equation

$$Z_{12}Z_{34} + Z_{13}Z_{42} + Z_{14}Z_{23} = 0,$$

where the coordinate Z_{ij} in $\mathbb{F}_p^4 \wedge \mathbb{F}_p^4$ represents the coordinate of the vector $z_i \wedge z_j$ of the fixed basis from above. The group G is thus given by selecting a subspace R in the 6-dimensional vector space $\mathbb{F}_p^4 \wedge \mathbb{F}_p^4$, and its Bogomolov multiplier may be identified as

$$\mathrm{B}_0(G) \cong \frac{R}{\langle \mathfrak{P} \cap R \rangle}.$$

To determine the intersection $\mathfrak{P} \cap R$, one can parametrize elements of R with a suitable basis and thus determine the quadratic form obtained by restricting $\mathfrak{P}$ to R. Determining whether $\mathrm{B}_0(G)$ is trivial then amounts to finding out if the solutions of the quadratic form $\mathfrak{P}|_R$ span the whole R.

***Example* 3.20** ([Sal84])**.** *Take G to be the p-group with $p > 2$ of nilpotency class 2 and exponent p, generated by the elements a, b, c, d subject to the relation $[a, b] = [c, d]$. Thus $|G| = p^9$. The commutator relation in G forms a line determined by the vector $a \wedge b - c \wedge d$ in the space $\mathbb{F}_p^4 \wedge \mathbb{F}_p^4$. The Plücker relation restricted to the subspace generated by the commutator relation is $\lambda(-\lambda) = 0$ for $\lambda \in \mathbb{F}_p$. There is only one trivial solution, whence $\mathrm{B}_0(G) \cong \langle a \wedge b - c \wedge d \rangle$. One can impose some additional commutator relations to G without spoiling the nontriviality of its Bogomolov multiplier.*

When considering p-groups of small orders, one can use classification results, especially as Bogomolov multipliers are invariant with respect to isoclinism. Determining the Bogomolov multiplier can, however, be quite involved. The smallest p-groups with nontrivial Bogomolov multipliers are of order 64 [CHKP08, CHKK10]. It is then easy to produce a myriad of examples by taking direct products. For odd primes p, the first nontrivial examples appear within groups of order p^5 [Bog87, Mor12]. In [HK11, HKK12, Mor12 p5] the authors prove that if G is a p-group of order p^5 with p odd, then $\mathrm{B}_0(G)$ is trivial if and only if G does not belong to a certain isoclinism family Φ_{10} appearing in the classification [Jam80]. Here is a concrete example of a group that belongs to this isoclinism family.

***Example* 3.21** ([HK11])**.** *Let $p > 3$ and take G to be the group generated by the elements $f_1, f_2, \ldots, f_5$ subject to the relations $f_5 \in Z(G)$, $f_1^p = f_5$, $f_i^p = 1$ for $2 \leq i \leq 5$, $[f_2, f_1] = f_3$, $[f_3, f_1] = f_4$, $[f_4, f_1] = [f_3, f_2] = f_5$, $[f_4, f_2] = [f_4, f_3] = 1$. This is a group of order p^5 and nilpotency class 4. It is therefore a group of maximal class. The authors of [HK11] show by means of a cohomological argument that its Bogomolov multiplier is nontrivial.*

These results enable us to provide another family of groups with trivial Bogomolov multipliers.

Proposition 3.22. *Let G be a group with $[G, G]$ of prime order. Then* $\mathrm{B}_0(G) = 0$.

Proof. Suppose the claim is false and take G to be a counterexample of minimal order. By minimality, G is a stem group and so $Z(G) = [G, G]$. Let $[G, G] = \langle z \rangle$ for some $z \in G$. The map $\mathrm{B}_0(G) \to \mathrm{B}_0(G/\langle z \rangle)$ is trivial. By Proposition 3.9, its kernel is equal to $J_z = \langle x \curlywedge y \mid [x, y] = z \rangle \cap \mathrm{B}_0(G)$. Thus by assumption, we have $J_z \neq 0$, and so there exist elements $a, b, c, d \in G$ with $a \curlywedge b, c \curlywedge d \in J_z$ and $a \curlywedge b \neq c \curlywedge d$. Therefore the relation $(a \curlywedge b)(c \curlywedge d)^{-1} \in \mathrm{B}_0(G)$ is nontrivial. By minimality, we must have $\langle a, b, c, d \rangle = G$. Note that $\exp G/[G, G] = p$, and so we can replace G by a group H isoclinic to it with the additional property $\exp H = p$. Note that we now have $|H| = |H/[H, H]| \cdot |[H, H]| \leq p^5$ and $\mathrm{B}_0(H) \neq 0$. This is only possible if H belongs to the isoclinism family Φ_{10} in [Jam80]. Since these groups are not of nilpotency class 2, we have reached a contradiction. ☐

3.2.7 Groups of maximal class

We now consider p-groups of maximal class in more detail. The structure of these groups can be understood best in terms of a certain series of normal subgroups [LGM02, Section 3]. Let G be a group of order p^n of maximal class, and suppose $n \geq 4$. There exists a chief series

$$G > P_1 > \ldots > P_n = 1$$

with $P_i = \gamma_i(G)$ for $i \geq 2$, and $P_1 = C_G(P_2/P_4)$, a 2-step centralizer. Set $P_i = 1$ for $i > n$. Note that $[P_i, P_j] \leq P_{i+j}$ for all $i, j \geq 1$. Pick any $s \in G \setminus P_1$ and $s_1 \in P_1 \setminus P_2$. The elements s and s_1 generate G. For $i \leq n$ define $s_i = [s_{i-1}, s]$. Then $s_i \in P_i \setminus P_{i+1}$ for all $i < n$. The groups $P_2, \ldots, P_n$ are the unique normal subgroups of G of index greater than p. The *degree of commutativity* of a p-group of maximal class is the largest integer ℓ with the property that $[P_i, P_j] \leq P_{i+j+\ell}$ for all $i, j \geq 1$ if P_1 is not abelian, and $\ell = n - 3$ otherwise. In our context, p-groups of maximal class with positive degree of commutativity will play an important role. Recall that if $n > p+1$, then G has positive degree of commutativity [LGM02, Theorem 3.3.5].

We will show the following characterization of nontriviality of Bogomolov multipliers.

Theorem 3.23. *Let G be a p-group of maximal class and order p^n. Then $\mathrm{B}_0(G)$ is trivial if and only if $[P_1, P_1] = [P_1, P_{n-2}]$.*

This implies that as long as $n > p+1$, and so in particular as $|G| \to \infty$, the group G has a nontrivial Bogomolov multiplier whenever it is not isoclinic to a group on the main line of the maximal coclass tree. Thus there are many p-groups with nontrivial Bogomolov multipliers. Our proof is explicit in pointing to a nonuniversal commutator relation in such a group G.

The proof will be split into two parts according to whether or not the Bogomolov multiplier is trivial. We first deal with the easier trivial case.

Theorem 3.24. *Let G be a p-group of maximal class with $[P_1, P_1] = [P_1, P_{n-2}]$. Then $\mathrm{B}_0(G)$ is trivial.*

Proof. If the subgroup P_1 is abelian, then G is abelian-by-cyclic and hence $\mathrm{B}_0(G)$ is trivial. So assume that P_1 is not abelian. The restriction $[P_1, P_1] = [P_1, P_{n-2}]$ gives $[P_1, P_1] = P_{n-1}$. Note that P_{n-1} is generated by the element s_{n-1} of order p. Moreover, since $[P_1, P_{n-2}] = P_{n-1}$ and $[P_2, P_{n-2}] = 1$, there exists a $\lambda \not\equiv 0 \bmod p$ with $[s_1, s_{n-2}] = s_{n-1}^{\lambda}$. The latter equality may be rewritten as $[s^{\lambda} s_1, s_{n-2}] = 1$. Expanding $1 = s^{\lambda} s_1 \curlywedge s_{n-2}$ in $G \curlywedge G$ gives

$$\begin{aligned}(s^{\lambda} \curlywedge s_{n-2})^{s_1}(s_1 \curlywedge s_{n-2}) &= (s^{\lambda}[s^{\lambda}, s_1] \curlywedge s_{n-2}s_{n-1}^{-\lambda})(s_1 \curlywedge s_{n-2}) \\ &= (s \curlywedge s_{n-2})^{\lambda}(s_1 \curlywedge s_{n-2}),\end{aligned}$$

therefore $(s_{n-2} \curlywedge s)^{\lambda} = (s_1 \curlywedge s_{n-2})$. Furthermore, pick any s_i, s_j, s_k, s_l in P_1 and assume that both of the elementary wedges $s_i \curlywedge s_j$ and $s_k \curlywedge s_l$ are nontrivial in $G \curlywedge G$. As $[P_1, P_1] = P_{n-1}$, both of the commutators $[s_i, s_j]$ and $[s_k, s_l]$ equal a power of s_{n-1}. Since $\mathrm{B}_0(P_1)$ is trivial by Proposition 3.22, there exists an $m > 0$ such that $(s_i \curlywedge s_j)(s_k \curlywedge s_l)^m \in \mathrm{B}_0(P_1)$ is trivial. The natural homomorphism $\mathrm{B}_0(P_1) \to \mathrm{B}_0(G)$ shows that $(s_i \curlywedge s_j)(s_k \curlywedge s_l)^m$ is also trivial in $G \curlywedge G$. Hence all the elementary wedges $s_i \curlywedge s_j$ are equal to a power of the nontrivial one $s_1 \curlywedge s_{n-2}$.

Now let w be an arbitrary element of $\mathrm{B}_0(G)$. For any $x, y \in P_1$ and $g, h \in G$, we have $[x \curlywedge y, g \curlywedge h] = [x, y] \curlywedge [g, h] = 1$ in $G \curlywedge G$, since $P_{n-1} = Z(G)$. Note also that $[s_i \curlywedge s, s_j \curlywedge s] = s_{i+1} \curlywedge s_{j+1}$. The element w may therefore be written in the form

$$w = \textstyle\prod_{i=1}^{n-2}(s_i \curlywedge s)^{\alpha_i} \cdot (s_1 \curlywedge s_{n-2})^{\beta}$$

for some integers α_i, β. Observe that

$$1 = s_i \curlywedge s^p = (s_i \curlywedge s^{p-1})(s_i \curlywedge s)^{s^{p-1}} = (s_i \curlywedge s)^p \cdot \textstyle\prod_{j>i}(s_j \curlywedge s)^{a_j} \cdot (s_1 \curlywedge s_{n-2})^b$$

for some a_j, b. We may thus assume that $0 \le \alpha_i < p$, and the same for β. Note that w belongs to $\mathrm{B}_0(G)$ if and only if we have $\prod_{i=1}^{n-2} s_{i+1}^{\alpha_i} \cdot [s_1, s_{n-2}]^{\beta} = 1$ in G. Collecting the left hand side in its normal form and comparing exponents gives $\alpha_i = 0$ for all $i \le n-3$ and $\alpha_{n-2} + \lambda\beta = 0$. We thus have $w = ((s \curlywedge s_{n-2})^{\lambda}(s_1 \curlywedge s_{n-2}))^{\beta}$, and so $w = 1$ by above. Hence $\mathrm{B}_0(G)$ is trivial, as required. □

We now turn to the nontrivial case. Note that this case is only possible when p is odd, since 2-groups of maximal class all have abelian P_1, see [LGM02, Corollary 3.3.4].

Theorem 3.25. *Let G be a p-group of maximal class with $[P_1, P_1] > [P_1, P_{n-2}]$. Then $\mathrm{B}_0(G)$ is nontrivial.*

Proof. The proof is done in four steps: reduction to the case $[P_1, P_1] = P_{n-1}$, construction of a central extension of G, checking for consistency to show that the extension is well-defined (this is the technical crux of the proof), and setting up a pairing that notices a nonuniversal commutator relation in G.

I. Let $[P_1, P_1] = P_m$ for some $m \le n-1$. For the sake of contradiction, suppose that $[P_1, P_{m-1}] = P_m$. Note that is only possible when the degree of commutativity of G is zero. Hence $n > 4$. Then either n or $n-1$ is odd and at least 5, and so G/P_{n-1} has positive degree of commutativity by [LGM02, Theorem 3.2.11]. Therefore $m = n-1$, contradicting the assumption $[P_1, P_1] > [P_1, P_{n-2}]$. Hence $P_m > [P_1, P_{m-1}]$. Using the Hopf-type formula for the Bogomolov multiplier, we obtain a surjective homomorphism $\mathrm{B}_0(G) \to \mathrm{B}_0(G/P_{m+1})$, since the subgroup P_{m+1} of G may be generated by commutators. Note that the group G/P_{m+1} is again of maximal class and satisfies the same condition on commutator subgroups as G does. In proving nontriviality of $\mathrm{B}_0(G)$, we may thus replace G by G/P_{m+1} and therefore assume that $[P_1, P_1] = P_{n-1}$ and $[P_1, P_{n-2}] = 1$. Note that the latter condition is equivalent, under the restriction $[P_1, P_1] = P_{n-1}$, to the group G having positive degree of commutativity.

II. The elements s and s_i with $1 \le i \le n-1$ form a polycyclic generating sequence of the group G. Let the corresponding power-commutator presentation be given in terms of the following relations:

$$\begin{gathered} s^p = s_{n-1}^{\lambda}, \quad s_i^p = \textstyle\prod_{k=i+1}^{n-1} s_k^{\mu_{i,k}} \text{ for } 1 \le i \le n-1, \\ [s_i, s] = s_{i+1} \text{ for } 1 \le i \le n-2, \quad [s_{n-1}, s] = 1, \\ [s_i, s_j] = s_{n-1}^{\nu_{i,j}} \text{ for } 1 \le j < i \le n-1. \end{gathered}$$

We introduce a new generator t, a so called *tail*, and tweak the above power-commutator presentation so as to form a central extension G^* of the cyclic group of order p by G, cf. Section 4.3 and [Nic93]. More precisely, the extension G^* is defined in terms of a polycyclic collector $\mathfrak{C}$ on the symbols s, s_i with $1 \le i \le n-1$, and an additional symbol t. We set the power and commutator relations of $\mathfrak{C}$ to be the following:

$$\begin{gathered} s^p = s_{n-1}^{\lambda}, \quad s_1^p = \textstyle\prod_{k=2}^{n-1} s_k^{\mu_{1,k}}, \quad s_i^p = \prod_{k=i+1}^{n-1} s_k^{\mu_{i,k}} \cdot t^{\mu_{i-1,n-2}} \text{ for } 2 \le i \le n-1, \\ [s_i, s] = s_{i+1} \text{ for } 1 \le i \le n-3, \quad [s_{n-2}, s] = s_{n-1} \cdot t, \quad [s_{n-1}, s] = 1, \\ [s_i, s_j] = s_{n-1}^{\nu_{i,j}} \text{ for } 1 \le j < i \le n-1, \quad [t, s] = [t, s_i] = 1 \text{ for } 1 \le i \le n-1, \quad t^p = 1. \end{gathered}$$

The next step is devoted to verifying that the collector $\mathfrak{C}$ is consistent and therefore gives a confluent polycyclic presentation of the group G^*.

III. There are four types of consistency checks to be made: associativity of the polycyclic generating sequence, right and left power associativities, and power commutativity, see [Nic93] for details. With each of these, we use the following notation. An element $g \in G^*$ is collected from the left according to $\mathfrak{C}$ into $[\![g]\!]$. The exponent of such a collected element at a symbol x is denoted by $[\![g]\!][x]$. Similarly, an element $g \in G$ is collected into its normal form $[\![g]\!]_G$. Note that the consistency checks that do not involve the generator s simply follow from the fact that the group G is given by a confluent polycyclic presentation, and that the tail t does not appear when applying only the commutator relations of $\mathfrak{C}$. This is due to the fact that we have $[\![s_{n-1}^p]\!] = [\![s_{n-1}^p]\!]_G t^{\mu_{n-2,n-2}} = 1$, since $s_{n-2}^p = 1$ in G by [LGM02, Proposition 3.3.2, Corollary 3.3.4].

III.1. ($s_j[\![s_i s]\!] = [\![s_j s_i]\!]s$) We have $s_{n-1}[\![s_i s]\!] = s s_i s_{i+1} s_{n-1} t^{\delta_{i,n-2}} = [\![s_{n-1}s_i]\!]s$, which covers the case when $j = n-1$. Similarly, we have $s_{n-2}[\![s_i s]\!] = s_{n-2} s s_i s_{i+1} = s s_i s_{i+1} s_{n-2} s_{n-1} t = [\![s_{n-2}s_i]\!]s$, dealing with the case when $j = n-2$. Finally, let $j < n-2$. Then $s_j[\![s_i s]\!] = s_j s s_i s_{i+1} = s s_j s_{j+1} s_i s_{i+1} = s[\![s_j s_{j+1} s_i s_{i+1}]\!]_G = [\![[\![s_j s_i]\!]_G s]\!]_G = [\![s_j s_i]\!]s$, where we have used that collecting the element $s_j s_{j+1} s_i s_{i+1}$ according to $\mathfrak{C}$ results in the same exponent vector as when collecting it in the group G.

III.2. ($[\![s_i^p]\!]s = s_i^{p-1}[\![s_i s]\!]$) This is clear when $i = n-1$. We have that $s_{n-3}^p \in P_{n-1}$ in G by [LGM02, Proposition 3.3.2, Corollary 3.3.4], and so the consistency check is also valid for $i = n-2$. Now let $i < n-2$. Then

$$s_i^{p-1}[\![s_i s]\!] = s(s_i s_{i+1})^p = s s_i^p s_{i+1}^p = s \prod_{k=i+1}^{n-1} s_k^{\mu_{i,k}} \prod_{k=i+1}^{n-1} s_{k+1}^{\mu_{i,k}} \cdot t^{\mu_{i-1,n-2}+\mu_{i,n-2}},$$

where we have used the equality $[s_i^p, s] = s_i^{-p}(s_i^p)^s = s_i^{-p}(s_i s_{i+1})^p = s_{i+1}^p$ in G to relate $[\![s_{i+1}^p]\!]$ with $[\![s_i^p]\!]$. At the same time, we have

$$[\![s_i^p]\!]s = \prod_{k=i+1}^{n-1} s_k^{\mu_{i,k}} s \cdot t^{\mu_{i-1,n-2}} = s \prod_{k=i+1}^{n-1} (s_k s_{k+1})^{\mu_{i,k}} \cdot t^{\mu_{i,n-2}+\mu_{i-1,n-2}}.$$

By validating this consistency check in the group G, we thus obtain the equality

$$\prod_{k=i+1}^{n-1} s_k^{\mu_{i,k}} \prod_{k=i+1}^{n-1} s_{k+1}^{\mu_{i,k}} = \prod_{k=i+1}^{n-1} (s_k s_{k+1})^{\mu_{i,k}}$$

in G. The consistency check now follows by applying the latter to the above, taking into account that using commutator relations in $\mathfrak{C}$ does not produce new appearances of t.

III.3. ($s_i[\![s^p]\!] = [\![s_i s]\!]s^{p-1}$) This is clear when $i \geq n-2$. Now let $i < n-2$, and put $k = n-2-i$. Modulo the exponent at s_{n-1}, we have

$$[\![s_i s]\!]s^{p-1} = \left[\!\!\left[\prod_{j=0}^{p} s_{i+j}^{\binom{p}{j}}\right]\!\!\right] \cdot t^{\sum_{j=0}^{p-1-k}\binom{k+j}{k}} = \left[\!\!\left[\prod_{j=0}^{k} s_{i+j}^{\binom{p}{j}}\right]\!\!\right] \cdot t^{\binom{p}{k+1}}.$$

Regarding collection modulo the exponents at s_{n-1}, we have

$$\left(\prod_{j=1}^{p} s_{i-1+j}^{\binom{p}{j}}\right)^{-1} s^{-1} \left[\!\!\left[\prod_{j=1}^{p} s_{i-1+j}^{\binom{p}{j}} s\right]\!\!\right] = \prod_{j=1}^{p} s_{i+j}^{\binom{p}{j}} \cdot t^{\delta_{i,n-1-p}}.$$

Invoking the previous consistency checks, we may collect $\prod_{j=1}^{p} s_{i-1+j}^{\binom{p}{j}}$ in $\mathfrak{C}$ before transferring s, which shows that

$$\left[\!\!\left[\prod_{j=1}^{p} s_{i+j}^{\binom{p}{j}}\right]\!\!\right][t] + \delta_{i,n-1-p} = \left[\!\!\left[\prod_{j=1}^{p} s_{i-1+j}^{\binom{p}{j}}\right]\!\!\right][s_{n-2}].$$

Note that we actually have $\prod_{j=1}^{p} s_{i-1+j}^{\binom{p}{j}} \in P_{n-1}$ in the group G by [LGM76, Proposition 3.1]. Whence

$$[\![s_i s]\!] s^{p-1} = \left[\!\!\left[\prod_{j=0}^{k} s_{i+j}^{\binom{p}{j}}\right]\!\!\right] \cdot t^{\binom{p}{k+1}} = s_i t^{-\delta_{i,n-1-p}+\binom{p}{k+1}} = s_i = s_i [\![s^p]\!]$$

modulo the exponent at s_{n-1}. Since the consistency check is valid in the group G, we in fact have $[\![s_i s]\!] s^{p-1} = s_i [\![s^p]\!]$ with respect to $\mathfrak{C}$, as required.

III.4. ($s[\![s^p]\!] = [\![s^p]\!]s$) This is straightforward, $s[\![s^p]\!] = s s_{n-1}^{\lambda} = s_{n-1}^{\lambda} s = [\![s^p]\!]s$.

IV. Finally, we verify that the tail has been attached in an appropriate manner so as to recognize a nonuniversal commutator relation in G. To begin with, since the group P_1 is nonabelian, there exist k, l such that $[s_k, s_l] = s_{n-1}^{\nu_{k,l}}$ with $\nu_{k,l} > 0$. This implies that we have $[s_{n-2}, s]^{\nu_{k,l}} [s_k, s_l]^{-1} = 1$ in the group G. We now construct a B_0-pairing $\phi\colon G \times G \to G^*$ that recognizes the latter relation as a nonuniversal one. To this end, let $\iota\colon G \to G^*$ be the natural mapping $s^\iota = s$ and $s_i^\iota = s_i$. Note that ι is not a homomorphism. The pairing ϕ is now defined as $\phi(x, y) = [x^\iota, y^\iota]$ for $x, y \in G$. We have $\phi(xy, z) = [(xy)^\iota, z^\iota] = [x^\iota y^\iota, z^\iota] = [(x^y)^\iota, (z^y)^\iota][y^\iota, z^\iota] = \phi(x^y, z^y)\phi(y, z)$, and similarly $\phi(x, yz) = \phi(x, z)\phi(x^z, y^z)$ and $\phi(x, x) = 1$ for all $x, y, z \in G$. Now suppose that $[x, y] = 1$ for some $x, y \in G$. If either x or y is central in G, then its corresponding image under ι is also central in G^*, and we have $\phi(x, y) = 1$. So assume that neither x nor y belong to P_{n-1}. If $x \notin P_1$, then $C_G(x) = \langle x, s_{n-1}\rangle$ by [Bla58], since $Z(G) < P_{n-2}$ and so x does not centralize P_{n-2}. This is the crucial point where we use the restriction $[P_1, P_{n-2}] = 1$. We therefore have $y = x^\alpha s_{n-1}^\beta$, which implies $\phi(x, y) = \phi(x, s_{n-1})^\beta = 1$, since s_{n-1} is central in G^*. Now let $x \in P_1$. A symmetric argument enables us to assume that $y \in P_1$. Writing both x and y in normal form with respect to the given polycyclic generating sequence in G as $x = \prod_{i=1}^{n-1} s_i^{u_i}$ and $y = \prod_{i=1}^{n-1} s_i^{v_i}$, we may rewrite $\phi(x, y)$ in terms of products of elementary terms $\phi(s_i, s_j)$ as follows:

$$\phi(x, y) = \prod_{i,j=1}^{n-1} \phi(s_i, s_j)^{u_i v_j} = \prod_{i,j=1}^{n-1} s_{n-1}^{u_i v_j \nu_{i,j}}.$$

Note that since $[x, y] = 1$, we have $\sum_{i,j} u_i v_j \nu_{i,j} = 0$, and therefore $\phi(x, y) = 1$. The mapping ϕ in therefore a B_0-pairing, and induces a homomorphism $\phi^*\colon G \curlywedge G \to G^*$ with $\phi^*(x \curlywedge y) = \phi(x, y)$ for all $x, y \in G$. Finally, observe that the element $v = (s_{n-2} \curlywedge s)^{\nu_{k,l}} (s_k \curlywedge s_l)^{-1}$ belongs to $\mathrm{B}_0(G)$, and $\phi^*(v) = t^{\nu_{k,l}}$ is nontrivial in G^*. This shows that the element v is itself nontrivial in $\mathrm{B}_0(G)$. The proof is complete. □

One of the nonuniversal commutator relations in a p-group of maximal class satisfying the condition $[P_1, P_1] > [P_1, P_{n-2}]$ may be read off from the proof of Theorem 3.25. Specifically, let $[P_1, P_1] = P_m$ for some m in such a group G. The commutator $[s_{m-1}, s]$ may be expanded as $[s_{m-1}, s] = \prod_{i,j}[s_i, s_j]$. Put $v = (s_{m-1} \curlywedge s)^{-1} \prod_{i,j} s_i \curlywedge s_j$ and note that by the proof of Theorem 3.25, the element v does not belong to the kernel of the natural homomorphism $\mathrm{B}_0(G) \to \mathrm{B}_0(G/P_{m+1})$. Therefore v is nontrivial in $\mathrm{B}_0(G)$.

Consider some special cases. When the degree of commutativity of G is positive, then the condition for $\mathrm{B}_0(G)$ to be trivial is simply that the maximal subgroup P_1 of G is abelian. For a given prime p, this implies that all p-groups of maximal class of large enough orders whose 2-step centralizer is non-abelian have non-trivial Bogomolov multipliers. With respect to isoclinism, this amounts to precisely the groups not isoclinic to a group on the main line of the coclass tree. Also, the cases $p = 2, 3$ are special. When $p = 2$, all the groups have abelian P_1, and therefore trivial Bogomolov multipliers. When $p = 3$, all the groups have positive degree of commutativity and $|[P_1, P_1]| \leq 3$, so we either have that P_1 is abelian, in which case $\mathrm{B}_0(G)$ is trivial, or $[P_1, P_1] = P_{n-1}$ and the degree is positive, in which case $\mathrm{B}_0(G)$ is nontrivial. Moreover, it follows from the proof of Theorem 3.24 that in this case, we have $\mathrm{B}_0(G) = \langle (s \curlywedge s_{n-1})^{\lambda} (s_i \curlywedge s_j) \rangle \cong C_3$ for some λ, i and j.

Note that the 2-groups of maximal class all have trivial Bogomolov multipliers. We exhibit examples of 2-groups of coclass 2 with non-trivial Bogomolov multipliers.

***Example* 3.26.** *Let $n \geq 7$ be a positive integer with $n = 2k + 3$ for some $k \geq 2$. Let*

$$G_n = \left\langle \; x_1, x_2, y \;\middle|\; \begin{array}{l} x_1^{2^{k+1}} = x_2^{2^k} = y^4 = 1, \\ [x_1, y] = x_2, \; [x_2, y] = x_1^{-2} x_2^{-2}, \; [x_1, x_2] = x_1^{2^k} \end{array} \; \right\rangle.$$

The group G_n is of order 2^n and class $n - 2$. Since $Z(G_n) = \langle [x_1, {}_{n-3}y] \rangle \cong C_2$, G_n is a stem group. Observe that $[G_n, G_n] = \langle x_1^2, x_2 \rangle \cong C_{2^k} \times C_{2^k}$. The curly exterior square $G_n \curlywedge G_n$ is therefore an abelian group, generated by the elements $x_1 \curlywedge x_2$, $x_1 \curlywedge y$ and $x_2 \curlywedge y$. Every element w of $G_n \curlywedge G_n$ may be written in the form $w = (x_1 \curlywedge x_2)^{\alpha} (x_1 \curlywedge y)^{\beta} (x_2 \curlywedge y)^{\gamma}$ for some integers α, β, γ. Note that such a w belongs to $\mathrm{B}_0(G_n)$ precisely when $x_1^{2^k \alpha} x_2^{\beta} (x_1^{-2} x_2^{-2})^{\gamma}$ is trivial in G_n. This is equivalent to $x_1^{2^k \alpha - 2\gamma} x_2^{\beta - 2\gamma} = 1$. Denoting $v = (x_1 \curlywedge x_2)(x_2 \curlywedge y)^{2^{k-1}}$, we thus have $\mathrm{B}_0(G) = \langle v \rangle$, where the order of v divides 2. Let us now show that the element v is in fact nontrivial in $G_n \curlywedge G_n$. To this end, observe that every element x of G_n can be written in the form $x \equiv x_1^{\alpha} y^{\beta}$ modulo $[G_n, G_n]$ for some $0 \leq \alpha < 2$ and $0 \leq \beta < 4$. Let $\phi \colon G_n \times G_n \to C_2 = \langle g \rangle$ be the mapping defined by

$$\phi(x_1^{\alpha_1} y^{\beta_1} d_1, x_1^{\alpha_2} y^{\beta_2} d_2) = g^{\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}},$$

where $d_1, d_2 \in [G_n, G_n]$. Note that this definition in fact represents the commutator map from $G_n/[G_n, G_n] \times G_n/[G_n, G_n]$ onto $[G_n, G_n]/\gamma_3(G_n)$. The map ϕ is bilinear in the exponent vectors of its two parameters. Suppose now that we have $[a, b] = 1$ for some $a = x_1^{\alpha_1} y^{\beta_1} d_1$ and $b = x_1^{\alpha_2} y^{\beta_2} d_2$. If $\beta_1 \equiv \beta_2 \equiv 0$ modulo 2, then $\phi(a, b) = 1$. Assume now that $\beta_1 \not\equiv 0$ modulo 2. Then $b \in C_{G_n}(a) = \langle a, x_1^{2^k} \rangle$, and therefore $\phi(a, b) = 1$. We have thus shown that the mapping ϕ is a B_0-pairing, hence it determines a unique homomorphism $\phi^* \colon G_n \curlywedge G_n \to C_2$ such that $\phi^*(a, b) = \phi(a, b)$ for all $a, b \in G_n$. As we have $\phi^*(v) = g$, the element v is nontrivial. Hence $\mathrm{B}_0(G_n) = \langle v \rangle \cong C_2$.

We record a simple corollary of Theorem 3.25 that shows how the result for maximal class and the groups from Example 3.26 produce groups of arbitrary coclass with nontrivial Bogomolov multipliers. This is essentially due to the fact that considering groups by isoclinism behaves well with respect to their nilpotency class, while coclass precisely complements this approach.

Corollary 3.27. *For every prime p and $c \geq 1$ ($c \geq 2$ for $p = 2$), there are infinitely many p-groups G of coclass c with* $\mathrm{B}_0(G) \neq 0$.

Proof. Let first p be odd. For a group H of maximal class with $\mathrm{B}_0(H) \neq 0$, set $G = H \times A$ with A an abelian group of order p^{c-1}. Then $\mathrm{B}_0(G) \cong \mathrm{B}_0(H)$ and G is of coclass c. For $p = 2$, apply the same argument to the group H in Example 3.26. □

We now turn to the more difficult task of explicitly determining the Bogomolov multiplier of a p-group of maximal class. We will do this for the subclass of groups in which P_1 is of nilpotency class 2 by translating determining $\mathrm{B}_0(G)$ to dealing with the much simpler commutator structure of P_1.

Theorem 3.28. *Let G be a p-group of maximal class with positive degree of commutativity and P_m abelian. Then $\mathrm{B}_0(G)$ is isomorphic to the coinvariants $(P_1 \curlywedge P_1)_G$.*

Note that by [LGM02, Theorem 3.4.11], a group G of order p^n (when $p \geq 5$) always has P_m abelian provided that $n \geq 6p - 24$.

In particular, Theorem 3.28 implies that the short exact sequence $1 \to \mathrm{B}_0(P_1) \to P_1 \curlywedge P_1 \to P_m \to 1$ of G-modules induces an exact sequence

$$\mathrm{B}_0(P_1)_G \to \mathrm{B}_0(G) \to C_p \to 1.$$

It follows that if G is metabelian with nonabelian P_1, then $\mathrm{B}_0(P_1) = 0$, and so $\mathrm{B}_0(G) \cong C_p$.

Note also that after considering the epimorphism $G \to G/P_{m+1}$, Theorem 3.25 may be deduced from Theorem 3.28.

The proof relies on observing the situation directly in a free presentation of G and referring to the Hopf-type formula for the Bogomolov multiplier.

Proof of Theorem 3.28. By [LGM02, Exercise 3.3(4)], the group G may be presented in the following manner. Let F be the free group on t and t_1. Denote $t_i = [t_{i-1}, t]$ for $2 \leq i$. Every element g of G has a normal form in terms of the generating set s and s_i for $1 \leq i \leq n-1$. For a word w of F, let $[\![w]\!]$ denote the word in t and t_i for $1 \leq i \leq n-1$ obtained by replacing s with t and s_i with t_i in the normal form of the element of G that is represented by the word w. Denote by $\rho(w) = w[\![w]\!]^{-1}$ the relator associated to w. Set

$$\mathcal{R}_0 = \{t_n\},\ \mathcal{R}_1 = \{\rho(t^p), \rho((tt_1)^p)\},\ \mathcal{R}_2 = \{\rho([t_{2i}, t_1]) \mid 1 \leq i \leq (p-1)/2\}.$$

Let R be the normal subgroup of F, generated by $\mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2$. Then F/R is a presentation of the group G. Finally, let $M = \langle\langle t_1 \rangle\rangle R$ be the maximal subgroup of F that corresponds to P_1.

Define λ to be the map

$$\lambda \colon M' \longrightarrow \frac{F' \cap R}{\langle \mathrm{K}(F) \cap R \rangle} \cong \mathrm{B}_0(G), \qquad w \longmapsto \rho(w)\langle \mathrm{K}(F) \cap R \rangle.$$

The rest of the proof is devoted to showing how λ induces the desired isomorphism between $(P_1 \curlywedge P_1)_G$ and $\mathrm{B}_0(G)$.

I. *We first claim that λ is a homomorphism.* To see this, first observe that since P_m is assumed to be abelian, we have $[M'\gamma_m(F), M'\gamma_m(F)] \leq R$. Now pick any $x, y \in M'$. Note that $[\![x]\!], [\![y]\!] \in \gamma_m(F)$. Hence

$$\lambda(x)\lambda(y) = x[\![x]\!]^{-1}y[\![y]\!]^{-1} \equiv xy([\![x]\!][\![y]\!])^{-1} \pmod{\langle \mathrm{K}(F) \cap R\rangle}.$$

Observe now that since $[\gamma_m(F), \gamma_m(F)] \leq \langle \mathrm{K}(F) \cap R\rangle$, every element of $\langle t_m, \dots, t_{n-1}\rangle$ may be rewritten as

$$\textstyle\prod_{i=m}^{n-1} t_i^{a_i} \equiv \left[\prod_{i=m}^{n-1} t_{i-1}^{a_i}, t\right] \pmod{\langle \mathrm{K}(F) \cap R\rangle}.$$

Thus $\langle t_m, \dots, t_{n-1}\rangle \cap R \leq \langle \mathrm{K}(F) \cap R\rangle$. Now, as $[\![x]\!][\![y]\!][\![xy]\!]^{-1} \in R \cap \langle t_m, \dots, t_{n-1}\rangle$, we conclude

$$\lambda(x)\lambda(y) = xy[\![xy]\!]^{-1} = \lambda(xy) \pmod{\langle \mathrm{K}(F) \cap R\rangle}.$$

II. *Let us now show that λ is surjective.* Consider the group $F/\langle R \cap \mathrm{K}(F)\rangle$. Its subgroup $R/\langle R \cap \mathrm{K}(F)\rangle$ is an abelian group that can be generated by the cosets of the elements of $\mathcal{R}_1 \cup \mathcal{R}_2$. Observe that $R/(R \cap F') \cong RF'/F'$ can be generated by the cosets of elements of $\mathcal{R}_1$. Moreover, the elements $t^p F'$ and $t_1^p F'$ form a base of the free abelian group RF'/F' of rank 2. Hence we have that the torsion group $(R \cap F')/\langle R \cap \mathrm{K}(F)\rangle$ is generated by the cosets of the elements of $\mathcal{R}_2$. Now note that $\mathcal{R}_2 \subseteq \rho(M')$. Therefore λ is indeed surjective.

III. *The homomorphism λ factors through $\langle \mathrm{K}(M) \cap R\rangle [M', t]$.* It is clear that $\langle \mathrm{K}(M) \cap R\rangle$ is contained in the kernel of λ. To see that the same holds for $[M', t]$, consider $\lambda(m^t)$ for some $m \in M'$. We have that $[\![m^t]\!] \equiv [\![m]\!]^t$ modulo $R \cap \langle t_m, \dots, t_{n-1}, t_n\rangle \leq \langle \mathrm{K}(F) \cap R\rangle$. Whence

$$\lambda(m^t) = m^t[\![m^t]\!]^{-1} \equiv m^t[\![m]\!]^{-t} \equiv m[\![m]\!]^{-1}[m[\![m]\!]^{-1}, t] \equiv \lambda(m) \pmod{\langle \mathrm{K}(F) \cap R\rangle},$$

proving our claim. There is thus an induced homomorphism

$$\bar{\lambda}\colon \frac{M'}{\langle \mathrm{K}(M) \cap R\rangle [M', t]} \cong (P_1 \curlywedge P_1)_G \longrightarrow \frac{F' \cap R}{\langle \mathrm{K}(F) \cap R\rangle} \cong \mathrm{B}_0(G).$$

IV. *Lastly, let us show that $\bar{\lambda}$ is injective.* To this end, let $w \in M'$ represent an element in $\ker \bar{\lambda}$. So $w[\![w]\!]^{-1} \in \langle \mathrm{K}(F) \cap R\rangle$. Write $w = [\![w]\!] \cdot \prod_i [x_i, y_i]$ for some $[x_i, y_i] \in R$. Collect all those indices i for which $x_i, y_i \in M$ into a set I. Upon replacing w by $w \prod_{i \in I}[x_i, y_i]^{-1}$, we may assume that we have $x_i \notin M$ and $y_i \in M$ for all indices i. Write $x_i = t^{a_i} m_i$ for some $1 \leq a_i < p$ and $m_i \in M$. Since $[x_i, y_i] \in R$, it follows that $y_i = t_{n-1}^{b_i} r_i$ for some $0 \leq b_i < p$ and $r_i \in R$. So $[x_i, y_i] = [t^{a_i} m_i, t_{n-1}^{b_i} r_i] \equiv [t^{a_i}, t_{n-1}^{b_i} r_i][m_i, t_{n-1}^{b_i} r_i]$ modulo $\langle \mathrm{K}(M) \cap R\rangle [M', t]$. Now $[m_i, t_{n-1}^{b_i} r_i] \in \langle \mathrm{K}(M) \cap R\rangle$, so that $[x_i, y_i] \equiv [t, t_{n-1}]^{a_i b_i}[t, r_i]^{a_i}$. We may thus write $w \equiv [\![w]\!] \cdot [t, t_{n-1}^a][t, r]$ for some $0 \leq a < p$ and $r \in R$. Since we are in a setting where $[M', t] \equiv 1$, it suffices to consider the element $r \in M$ modulo M'. Now, the group M/M' can be generated by the elements

t^p and t_i for $i \geq 1$, and it follows from this that we may write $w \equiv \prod_{i \geq m} t_i^{c_i}$ for some integers c_i.

Note that the image of the group $\langle t_i \mid i \geq m \rangle$ in $M/\langle \mathrm{K}(M) \cap R \rangle$ is abelian, and so it is a quotient of the free abelian group generated by the elements t_i for $i \geq m$. Moreover, the group M/M' is the quotient of the free abelian group generated by the elements t^p and t_i for $i \geq 1$ subject only to the relations

$$\prod_{i=1}^{p} t_{j+i}^{\binom{p}{i}} \equiv 1 \pmod{M'}$$

for all $j \geq 1$. These arise from expanding $[t_j, t^p] \in M'$. As the element $w \equiv \prod_{i \geq m} t_i^{c_i}$ belongs to M', it can therefore be written as a product of some powers of elements of the form $\prod_{i=1}^{p} t_{j+i}^{\binom{p}{i}}$ for some $j \geq m$. Now, observe that

$$\prod_{i=1}^{p} t_{j+i}^{\binom{p}{i}} \equiv [t_j, t^p] \equiv 1 \pmod{\langle \mathrm{K}(M) \cap R \rangle}$$

for all $j \geq m-1$. Therefore $w \equiv 1$ in the domain of $\bar{\lambda}$ and the proof is complete. $\square$

Following the proof of Theorem 3.28, we provide some examples of p-groups of maximal class (for $p > 3$) for which the structure of their Bogomolov multipliers may be explicitly determined.

To do this, we recall that the structure of a p-group G of maximal class with P_1 of nilpotency class 2 can be given in terms of the ring of integers in the p-th cyclotomic number field $\mathcal{O}$. So $\mathcal{O} = \mathbb{Z}[\theta]/(1 + \theta + \cdots + \theta^{p-1})$, where θ is a primitive complex p-th root of unity. Denote $\kappa = \theta - 1$ and let $\mathfrak{p} = (\kappa)$. There is an action of $\mathcal{O}$ on P_m with θ acting via conjugation by s. By [LGM02, Lemma 8.2.1], there is an $\mathcal{O}$-module isomorphism between P_i/P_{i+j} and $\mathcal{O}/\mathfrak{p}^j$, induced by the map

$$\mathcal{O} \to P_i/P_{i+j}, \quad \sum_u a_u \kappa^u \mapsto \prod_u s_{i+u}^{a_u}.$$

The commutator structure of P_1 can thus be understood in terms of the homomorphism

$$\alpha \colon \mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1} \to \mathcal{O}/\mathfrak{p}^{n-m} \cong P_m.$$

This is in fact a homomorphism of $\langle \theta \rangle = C_p$ modules. Set

$$\mathrm{K}\,\alpha = \langle \ker \alpha \cap \{\text{elementary wedges in } \mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1}\} \rangle.$$

Now consider the induced epimorphism

$$\alpha_{C_p} \colon (\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1})_{C_p} \to (\mathcal{O}/\mathfrak{p}^{n-m})_{C_p} \cong \mathcal{O}/\mathfrak{p} \cong P_m/P_{m+1} \cong C_p$$

obtained by factoring out the action of θ. Correspondingly, there is the induced kernel

$$\mathrm{K}\,\alpha_{C_p} = \langle \ker \alpha_{C_p} \cap \{\text{image of elementary wedges in } (\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1})_{C_p}\} \rangle.$$

Notice that
$$\frac{(\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1})_{C_p}}{\mathrm{K}\,\alpha_{C_p}} \cong \left(\frac{\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1}}{\mathrm{K}\,\alpha}\right)_{C_p}$$
by right-exactness of coinvariants. We make the following identification:
$$P_1 \curlywedge P_1 = \frac{P_1/P_m \wedge P_1/P_m}{\langle xP_m \wedge yP_m \mid [x,y]=1\rangle} \cong \frac{\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1}}{\mathrm{K}\,\alpha}$$
Now, to provide concrete examples, we show that by carefully selecting the map α, which in turn determines the group G, one may achieve that the image of the map $\mathrm{K}\,\alpha_{C_p}$ in $(\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1})_{C_p}$ is trivial. Based on the above identification, this amounts to constructing groups G with $\mathrm{B}_0(G) \cong (\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1})_{C_p}$. Such a commutator structure will therefore produce groups whose Bogomolov multipliers will have largest possible rank and exponent for the given values of n and m. Furthermore, essentially the same argument will deal with quotients of such extreme groups. The construction we give below covers this more general case.

Fix any $m \geq 4$ and set $l = m-3$. The number l will be the degree of commutativity of the constructed group. Now pick any n satisfying $m < n \leq 2m-2$. Set $\mu = n-m+2$, so that $2 < \mu \leq m$. Let g be a primitive root modulo p and pick an integer a so that $a \equiv (g+1)^{-1} \pmod{p}$. It is here that we need $p > 3$. In the case when $a > (p-1)/2$, replace a by $1-a$, so that in the end, $2 \leq a \leq (p-1)/2$. Now define $\alpha\colon \mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1} \to \mathcal{O}/\mathfrak{p}^{n-m}$ by the rule
$$\alpha(x \wedge y) = \kappa^{-1} \cdot (\sigma_a(x)\sigma_{1-a}(y) - \sigma_a(y)\sigma_{1-a}(x))$$
for $x, y \in \mathcal{O}/\mathfrak{p}^{m-1}$. Here, σ_a is the automorphism of $\mathcal{O}/\mathfrak{p}^{m-1}$ which maps θ to θ^a. This corresponds to the map induced by $\kappa^{-1}S_a$ in [LGM02, Theorem 8.3.1]. Set $u_a = (\theta^a - 1)/\kappa \in \mathcal{O}^*$. Then
$$\alpha(\kappa^i \wedge \kappa^j) = \mathrm{sgn}(i-j)\kappa^{i+j-1}(u_a u_{1-a})^{\min\{i,j\}}(u_a^{|i-j|} - u_{1-a}^{|i-j|}) \in \mathfrak{p}^{i+j-1}.$$
Observe that $u_a^{|i-j|} - u_{1-a}^{|i-j|} \equiv a^{|i-j|} - (1-a)^{|i-j|} \pmod{\mathfrak{p}}$. This element belongs to $\mathfrak{p}$ if and only if we have $(a^{-1}-1)^{|i-j|} \equiv 1 \pmod{p}$. By our choice of a, this occurs precisely when $i \equiv j \pmod{p-1}$. The commutator map α therefore satisfies $\alpha(\kappa^i \wedge \kappa^j) \in \mathfrak{p}^{i+j-1} \backslash \mathfrak{p}^{i+j}$ whenever $i \not\equiv j \pmod{p-1}$.

Invoking [LGM02, Theorem 8.2.7], there is a p-group G of maximal class of order p^n whose commutator structure is described by the map α given above. In terms of the P_i-series of G, the above discussion shows that we have $[P_i, P_j] = P_{i+j+l}$ for all $i, j \geq 1$ that satisfy $i \not\equiv j \pmod{p-1}$.

This highly restricted commutator structure enables us to completely understand commuting pairs of G.

Lemma 3.29. *Let $x \in P_i \backslash P_{i+1}$. Then $C_{P_i}(x) = \langle x, P_{i+j} \rangle$, where $j = \max\{n-2i-l, 1\}$.*

Proof. Clearly the right hand side centralizes x. Conversely, suppose that $y \in P_k \backslash P_{k+1}$ for some $k > i$ and $[x,y] = 1$. Assume that $y \notin \langle x \rangle$. If $k \equiv i \pmod{p-1}$, then $y = x^r z$

for some $r > 0$ and $z \in P_{k'} \backslash P_{k'+1}$ with $[z, x] = 1$ and $k' \not\equiv i \pmod{p-1}$. In this case, replace y by z and k by k', so that we may assume $k \not\equiv i \pmod{p-1}$. Now, since $[P_i, P_k] = P_{i+k+l}$ and $[P_{i+1}, P_k][P_i, P_{k+1}] \leq P_{i+k+1+l}$, it follows that $P_{i+k+l} = P_{i+k+1+l}$, which is only possible when $i + k + l \geq n$. □

In particular, note that $Z(P_1) \geq P_\mu \geq P_m$ in the group G. Transferring to the C_p-module $\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1}$, we thus have that the elementary wedges in $\mathfrak{p}^{\mu-1}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1}$ are all contained in $\mathrm{K}\,\alpha$. Using Lemma 3.29 more precisely, we now show that wedges that arise from commuting pairs are, modulo the action of C_p, nothing but the latter.

Lemma 3.30. $\mathrm{K}\,\alpha_{C_p} = (\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathfrak{p}^{\mu-1}/\mathfrak{p}^{m-1}) + [\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1}, C_p]$.

Proof. Let $x \wedge y \in \mathrm{K}\,\alpha$ for some $x, y \in \mathcal{O}/\mathfrak{p}^{m-1}$. Suppose that x corresponds to an element in $P_i \backslash P_{i+1}$ and y to an element in $P_j \backslash P_{j+1}$ with $i \leq j$. We will prove that $x \wedge y$ is equivalent to an element of the submodule $\mathfrak{p}^{\mu-1}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/p^{m-1}$ modulo $[\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1}, C_p]$ by induction on i.

If $i \equiv j \pmod{p-1}$, then as in the proof of Lemma 3.29, we may write $y = x^r z$ with $z \in P_{j'} \backslash P_{j'+1}$ and $j' \not\equiv i \pmod{p-1}$. Then $x \wedge y = x \wedge z$, so we may without loss of generality assume that $i \not\equiv j \pmod{p-1}$. By the lemma, we then have than $i+j+l \geq n$. If $i = 1$, this implies that $j \geq n - l - 1 = \mu$, whence $x \wedge y \in \mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathfrak{p}^{\mu-1}/\mathfrak{p}^{m-1}$. This is the base for the induction. Suppose now that $i > 1$. Then $x = \kappa\tilde{x}$ for some $\tilde{x} \in \mathcal{O}/\mathfrak{p}^{m-1}$ corresponding to a group element in $P_{i-1} \backslash P_i$. Observe that

$$\tilde{x} \wedge \kappa y + x \wedge y + x \wedge \kappa y = \kappa(\tilde{x} \wedge y) \in [\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1}, C_p].$$

Note that κy corresponds to a group element in P_{j+1}, and therefore $\tilde{x} \wedge \kappa y$ and $x \wedge \kappa y$ both belong to $\mathrm{K}\,\alpha$. Using this reasoning, we show our claim by reverse induction on j. When $j \geq \mu$, it is clear that $x \wedge y \in \mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathfrak{p}^{\mu-1}/\mathfrak{p}^{m-1}$. Assume now that $j < \mu$. By induction, both $\tilde{x} \wedge \kappa y$ (since $\tilde{x}$ belong to a higher level) and $x \wedge \kappa y$ (since κy belongs to a lower level) are contained in $\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathfrak{p}^{\mu-1}/\mathfrak{p}^{m-1}$ modulo $[\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1}, C_p]$. An application of (3.2.7) then implies that the same holds for $x \wedge y$, as claimed. □

The above gives that

$$\mathrm{B}_0(G) = \frac{(\mathcal{O}/\mathfrak{p}^{m-1} \wedge \mathcal{O}/\mathfrak{p}^{m-1})_{C_p}}{\mathrm{K}\,\alpha_{C_P}} = (\mathcal{O}/\mathfrak{p}^{\mu-1} \wedge \mathcal{O}/\mathfrak{p}^{\mu-1})_{C_p}.$$

Finally, a structure description of the group $(\mathcal{O}/\mathfrak{p}^{\mu-1} \wedge \mathcal{O}/\mathfrak{p}^{\mu-1})_{C_p}$ may be read off of the explicit C_p-module decomposition of $\mathcal{O}/\mathfrak{p}^{\mu-1} \wedge \mathcal{O}/\mathfrak{p}^{\mu-1}$ into a direct sum of cyclic submodules as given in [LGM78, Theorem 8.13]. We have thus proved the following.

Theorem 3.31. *Assume $p \geq 5$, let $m \geq 4$ and let n satisfy $m \leq n \leq 2m-2$. Write $n - m + 1 = x(p-1) + y$ for some $x \geq 0$ and $p - 1 > y \geq 0$. Then there exist p-groups of maximal class of order p^n with Bogomolov multipliers isomorphic to*

$$C_{p^{x+1}}^{\lfloor y/2 \rfloor} \times C_{p^x}^{(p-1)/2 - \lfloor y/2 \rfloor}.$$

Consider some special cases. When n is chosen so that $n \equiv m-1 \pmod{p-1}$, we obtain a group G with $\mathrm{B}_0(G)$ homocyclic of rank $(p-1)/2$ and exponent $p^{(n-m+1)/(p-1)}$. Further selecting $n \approx 2m$, we have the property $\exp \mathrm{B}_0(G) \approx \sqrt{\exp G}$. Consider now the option $n = m+1$. In this case, we obtain groups that are immediate descendants of groups on the main line of the maximal class tree. Their Bogomolov multipliers are C_p. In the very special case when $m = 4$, we obtain the known groups of order p^5 with nontrivial Bogomolov multipliers. Another extreme option is picking $n = 2m-2$. In this case, we have $n - m + 1 = m - 1$, so by varying m, the groups exhaust all the possibilities for the Bogomolov multiplier, depending on the value $m-1 \pmod{p-1}$. Finally, consider the option of selecting consecutive values $n = m+1, m+2, \dots, 2m-2$. In terms of the constructed groups, this corresponds to a path in the maximal class tree, starting from an immediate descendant of a group on the main line (which is of order p^m) and going deeper into the branch. In this process, the value $n-m+1$ grows one by one, so that the corresponding Bogomolov multipliers grow in size by p on each second step, starting with C_p for the group closest to the main line. The growth is "staircase"-like, consecutively increasing the orders of the generators by a factor of p on each second step.

By developing the theory further, we will show in Corollary 4.31 that the above rank of the Bogomolov multiplier is in fact largest possible for p-groups of maximal class.

4

Unraveling relations

In parallel to the classical theory of central extensions of groups, we develop a version for extensions that preserve commutativity. Thus we begin with a group Q and wish to understanding how to produce extensions G of Q that preserve commuting pairs. A cohomological object is introduced and it is shown that the Bogomolov multiplier is a universal object parameterizing such extensions of a given group. We also provide several characterizations of these extensions and prove that they are closed under isoclinism. Maximal and minimal extensions are inspected thoroughly and a theory of covers is developed. The latter is used to give an effective algorithm for computing Bogomolov multipliers. Lastly, we inspect the structure of groups that are minimal with respect to possessing a nonuniversal commutator relation.

This chapter is based on [JM13 arxiv, JM14 GAP, JM14 128, JM15, JM, Mor12].

4.1 Commutativity preserving extensions

4.1.1 CP extensions

Let Q be a group and N a Q-module. Denote by $e = (\chi, G, \pi)$ the extension

$$1 \longrightarrow N \xrightarrow{\chi} G \xrightarrow{\pi} Q \longrightarrow 1$$

of N by Q. Following [Mor12], we say that e is a *CP extension* if commuting pairs of elements of Q have commuting lifts in G.

Lemma 4.1. *The class of CP extensions is closed under equivalence of extensions.*

Proof. Let

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\mu_1"] \arrow[d, equal] & G_1 \arrow[r, "\epsilon_1"] \arrow[d, "\theta"] & Q \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & N \arrow[r, "\mu_2"] & G_2 \arrow[r, "\epsilon_2"] & Q \arrow[r] & 1
\end{tikzcd}$$

be equivalent extensions with abelian kernel. Suppose that G_1 is a CP extension of N by Q. Choose $x_1, x_2 \in Q$ with $[x_1, x_2] = 1$. Then there exist $e_1, e_2 \in G_1$ such that $[e_1, e_2] = 1$ and $\epsilon_1(e_i) = x_i$, $i = 1, 2$. Take $\bar{e}_i = \theta(e_i)$. Then $[\bar{e}_1, \bar{e}_2] = 1$ and $\epsilon_2(\bar{e}_i) = x_i$. This proves that G_2 is a CP extension of N by Q. $\square$

Up to equivalence, all the information on CP extensions can be encoded in a cohomological object. We refer to [Bro82] for an account on the theory of group extensions.

Let us first recall some standard definitions. Let Q and S be groups, and suppose that Q acts on S via $(x, y) \mapsto {}^{x}y$, where $x \in Q$ and $y \in S$. A map $\partial : Q \to S$ is a *derivation* (or *1-cocycle*) *from* Q *to* S if $\partial(xy) = {}^{x}\partial(y)\partial(x)$ for all $x, y \in Q$. Let N be a Q-module and fix $a \in N$. The map $\partial_a : Q \to N$, given by $\partial_a(g) = ga - a$, is a derivation. It is called an *inner derivation.*

A cocycle $\omega \in \mathrm{Z}^2(Q, N)$ is said to be a *CP cocycle* if for all commuting pairs $x_1, x_2 \in Q$ there exist $a_1, a_2 \in N$ such that

$$\omega(x_1, x_2) - \omega(x_2, x_1) = \partial_{a_1}(x_1) + \partial_{a_2}(x_2). \tag{4.1}$$

Denote by $\mathrm{Z}^2_{\mathrm{CP}}(Q, N)$ the set of all CP cocycles in $\mathrm{Z}^2(Q, N)$.

Proposition 4.2. $\mathrm{Z}^2_{\mathrm{CP}}(Q, N)$ *is a subgroup of* $\mathrm{Z}^2(Q, N)$ *containing* $\mathrm{B}^2(Q, N)$*.*

Proof. It is clear that $\mathrm{Z}^2_{\mathrm{CP}}(Q, N)$ is a subgroup of $\mathrm{Z}^2(Q, N)$. Now let $\beta \in \mathrm{B}^2(Q, N)$. Then there exists a function $\phi : Q \to N$ such that

$$\beta(x_1, x_2) = x_1\phi(x_2) - \phi(x_1x_2) + \phi(x_1)$$

for all $x_1, x_2 \in Q$. Suppose that these two elements commute. Then $\beta(x_1, x_2) - \beta(x_2, x_1) = \partial_{\phi(x_2)}(x_1) + \partial_{-\phi(x_1)}(x_2)$, hence $\beta \in \mathrm{Z}^2_{\mathrm{CP}}(Q, N)$. $\square$

Now define

$$\mathrm{H}^2_{\mathrm{CP}}(Q, N) = \mathrm{Z}^2_{\mathrm{CP}}(Q, N) / \mathrm{B}^2(Q, N).$$

This is a subgroup of the ordinary cohomology group $\mathrm{H}^2(Q, N)$.

***Example* 4.3.** *Let* Q *be an abelian group and* N *a trivial* Q*-module. Then* $\mathrm{H}^2_{\mathrm{CP}}(Q, N)$ *coincides with* $\mathrm{Ext}(Q, N)$*.*

Proposition 4.4. *Let* N *be a* Q*-module. Then the equivalence classes of CP extensions of* N *by* Q *are in bijective correspondence with the elements of* $\mathrm{H}^2_{\mathrm{CP}}(Q, N)$*.*

Proof. Let $e = (\chi, G, \pi)$ be an extension of N by Q. Let $\omega : Q \times Q \to N$ be a corresponding 2-cocycle. Then e is equivalent to the extension

$$1 \longrightarrow N \longrightarrow Q[\omega] \xrightarrow{\epsilon} Q \longrightarrow 1\ ,$$

where $Q[\omega]$ is, as a set, equal to $N \times Q$, and the operation is given by $(a, x)(b, y) = (a + xb + \omega(x, y), xy)$, and $\epsilon(a, x) = x$. By Lemma 4.1 it suffices to show that the latter extension is CP if and only if $\omega \in \mathrm{Z}^2_{\mathrm{CP}}(Q, N)$. Let $x, y \in Q$ commute and let (a, x) and (b, y) be lifts of x and y in $Q[\omega]$. Then (a, x) and (b, y) commute if and only if $\omega(x, y) - \omega(y, x) = (y - 1)a - (x - 1)b = \partial_a(y) + \partial_{-b}(x)$. Thus the existence of commuting lifts of x and y is equivalent to $\omega \in \mathrm{Z}^2_{\mathrm{CP}}(Q, N)$. $\square$

***Example* 4.5.** *Let Q be a group in which for every commuting pair x, y the subgroup $\langle x, y\rangle$ is cyclic. In the case of finite groups, it is known [Bro82, Theorem VI.9.5] that such groups are precisely the groups with periodic cohomology, and this further amounts to Q having cyclic Sylow p-subgroups for p odd, and cyclic or quaternion Sylow p-subgroups for $p = 2$. Infinite groups with this property include free products of cyclic groups, cf. [KS58]. Given such a group Q, it is clear that every commuting pair of elements in Q has a commuting lift. Thus every extension of Q is CP, and so $\mathrm{H}^2(Q, N) = \mathrm{H}^2_{\mathrm{CP}}(Q, N)$ for any Q-module N.*

***Example* 4.6.** *Taking the simplest case $Q = C_p$ in the previous example, we see that every extension of a group by C_p is CP. Thus in particular, every finite p-group can be viewed as being composed from a sequence of CP extensions.*

***Example* 4.7.** *There are many examples of extensions that are not CP. One may simply take as G a group of nilpotency class 2 and factor by a subgroup generated by a non-trivial commutator. In fact, in the case when the extension is central, it is more difficult to find examples of extensions that are CP. We will focus on inspecting central CP extensions in the following section. Consider now only extensions that are not central. Some small examples of extensions which fail to be CP are easily produced by taking a non-trivial action of a non-cyclic abelian group on an elementary abelian group. We give a concrete example. Take $Q = \langle x_1\rangle \times \langle x_2\rangle$ to be an elementary abelian p-group of rank 2, and let it act on $N = \langle a_1\rangle \times \langle a_2\rangle \times \langle a_3\rangle$, an elementary abelian p-group of rank 3, via the following rules:*

$$a_1^{x_1} = a_1,\ a_2^{x_1} = a_2,\ a_3^{x_1} = a_3,\ a_1^{x_2} = a_1,\ a_2^{x_2} = a_1a_2,\ a_3^{x_2} = a_3.$$

Thus N is a Q-module. Now construct an extension G corresponding to this action by specifying $x_2^{x_1} = x_2a_3$. This extension is not CP because the commuting pair x_1, x_2 in Q does not have a commuting lift in G.

4.1.2 Central CP extensions

From now on, we focus on a special type of CP extensions – those with central kernel. In terms of the cohomological interpretation, these correspond to the case when the relevant module is trivial.

The fundamental result here is a CP version of the Universal Coefficient Theorem. In other words, there exists a universal cohomological object that parameterizes all central CP extensions. We show that this object is the Bogomolov multiplier.

Theorem 4.8. *Let N be a trivial Q-module. Then there is a split exact sequence*

$$0 \longrightarrow \mathrm{Ext}(Q^{\mathrm{ab}}, N) \xrightarrow{\psi} \mathrm{H}^2_{\mathrm{CP}}(Q, N) \xrightarrow{\tilde{\varphi}} \mathrm{Hom}(\mathrm{B}_0(Q), N) \longrightarrow 0\ , \tag{4.2}$$

where the maps ψ and $\tilde{\varphi}$ are induced by the Universal Coefficient Theorem.

Proof. By the Universal Coefficient Theorem, we have a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(Q^{\mathrm{ab}}, N) \xrightarrow{\psi} \mathrm{H}^2(Q, N) \xrightarrow{\varphi} \mathrm{Hom}(\mathrm{M}(Q), N) \longrightarrow 0 . \tag{4.3}$$

Let $[\omega]$ belong to $\mathrm{Ext}(Q^{\mathrm{ab}}, N)$. Then [BT82] the map ψ can be described as $\psi([\omega]) = [\omega \circ (\mathrm{ab} \times \mathrm{ab})]$, where $\mathrm{ab} : Q \to Q^{\mathrm{ab}}$. If $x, y \in Q$ commute, then $\psi([\omega])(x, y) = \omega(x[Q, Q], y[Q, Q]) = \omega(y[Q, Q], x[Q, Q]) = \psi([\omega])(y, x)$, therefore ψ maps the group $\mathrm{Ext}(Q^{\mathrm{ab}}, N)$ into $\mathrm{H}^2_{\mathrm{CP}}(Q, N)$. The map φ can be described as follows. Suppose that $[\omega] \in \mathrm{H}^2(Q, N)$ represents a central extension

$$0 \longrightarrow N \longrightarrow \tilde{Q} \xrightarrow{\pi} Q \longrightarrow 1 . \tag{4.4}$$

Let $z = \prod_i (x_i \wedge y_i) \in \mathrm{M}(Q)$, that is, $\prod_i [x_i, y_i] = 1$. Choose $\tilde{x}_i, \tilde{y}_i \in \tilde{Q}$ such that $\pi(\tilde{x}_i) = x_i$ and $\pi(\tilde{y}_i) = y_i$. Define $\tilde{z} = \prod_i [\tilde{x}_i, \tilde{y}_i]$. Clearly $\tilde{z} \in N$, and it can be verified that the map φ is well defined by the rule $\varphi([\omega]) = (z \mapsto \tilde{z})$.

Suppose now that $[\omega]$ belongs to $\mathrm{H}^2_{\mathrm{CP}}(Q, N)$. Let z belong to $\mathrm{M}_0(Q)$. Then z can be written as $z = \prod_i (x_i \wedge y_i)$, where $[x_i, y_i] = 1$ for all i. Since the extension (4.4) is a central CP extension, we can choose commuting lifts $(\tilde{x}_i, \tilde{y}_i)$ of the commuting pairs (x_i, y_i). By the above definition, $\tilde{z} = 0$, hence φ is trivial when restricted to $\mathrm{M}_0(Q)$. Thus φ induces an epimorphism $\tilde{\varphi} : \mathrm{H}^2_{\mathrm{CP}}(Q, N) \to \mathrm{Hom}(\mathrm{B}_0(Q), N)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}^2(Q, N) \arrow[r, "\varphi"] & \mathrm{Hom}(\mathrm{M}(Q), N) \\
\mathrm{H}^2_{\mathrm{CP}}(Q, N) \arrow[u, "\iota"] \arrow[r, "\tilde{\varphi}"] & \mathrm{Hom}(\mathrm{B}_0(Q), N) \arrow[u, "\rho^*"']
\end{tikzcd}$$

Here the map ρ^* is induced by the canonical epimorphism $\rho : \mathrm{M}(Q) \to \mathrm{B}_0(Q)$. Therefore it follows that $\ker \tilde{\varphi} = \ker \varphi|_{\mathrm{im}\, \iota} = \mathrm{im}\, \psi$. This shows that the sequence (4.2) is exact. Furthermore, the splitting of the sequence (4.3) yields that the sequence (4.2) is also split. This proves the result. □

Here is a sample application the above theorem. Recall that Schur's theory of covering groups originally arised in the context of projective representations, cf. [Sch07]. Schur showed that there is a natural correspondence between the elements of $\mathrm{H}^2(Q, \mathbb{C}^\times)$ and projective representations of Q. To every projective representation $\rho\colon Q \to \mathrm{GL}(V)$ one can associate a cocycle $\alpha \in \mathrm{Z}^2(Q, \mathbb{C}^\times)$ via the rule $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$ for every $x, y \in Q$. Projectively equivalent representations induce cohomologous cocycles, and a cocycle is a coboundary if and only if the representation is equivalent to a linear representation. It is readily verified that CP extensions integrate well into this setting.

Proposition 4.9. *Projective representations $\rho\colon Q \to \mathrm{GL}(V)$ with the property that*

$$[\rho(x_1), \rho(x_2)] = 1 \quad \textit{whenever} \quad [x_1, x_2] = 1$$

correspond to cohomological classes of CP cocycles $\alpha \in \mathrm{Z}^2(Q, \mathbb{C}^\times)$, i.e., elements of $\mathrm{B}_0(Q)$.

In particular, if $\mathrm{B}_0(Q)$ is trivial, every projective representation of Q which preserves commutativity is similar to a linear representation. Such maps have been studied in detail in other algebraic structures, see [Šem08] for a survey. A loose connection can be made along the following lines. Let $\rho\colon Q \to S$ be a set-theoretical map from Q to a group S such that $\rho(1) = 1$ and the induced map $\rho\colon Q \to S/Z(S)$ is a homomorphism. We may thus write $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$ for some function $\alpha\colon Q \times Q \to Z(S)$. In view of the associativity of multiplication, α is in fact a $Z(S)$-valued 2-cocycle. As above, such maps ρ correspond to elements of $\mathrm{H}^2_{\mathrm{CP}}(Q, Z(S))$.

Next, we give a simple criterion for determining whether or not a given central extension is CP. This result will later be used repeatedly.

Proposition 4.10. *Let*

$$e: \quad 1 \longrightarrow N \xrightarrow{\chi} G \xrightarrow{\pi} Q \longrightarrow 1$$

be a central extension. Then e is a CP extension if and only if $\chi(N) \cap \mathrm{K}(G) = 1$.

Proof. Denote $M = \chi(N)$. Suppose that $M \cap \mathrm{K}(G) = 1$. Choose $x, y \in Q$ with $[x, y] = 1$. We have $x = \pi(g)$ and $y = \pi(h)$ for some $g, h \in G$. Then $\pi([g, h]) = 1$, hence $[g, h] \in M \cap \mathrm{K}(G) = 1$. Thus g and h are commuting lifts of x and y, respectively.

Conversely, suppose that e is a CP central extension. Choose $[g, h] \in M \cap K(G)$. By assumption, there exists a commuting lift $(g_1, h_1) \in G \times G$ of the commuting pair $(\pi(g), \pi(h))$. We can thus write $g_1 = ga$, $h_1 = hb$, where $a, b \in M$. It follows that $1 = [g_1, h_1] = [ga, hb] = [g, h]$, hence $M \cap \mathrm{K}(G) = 1$. ☐

It is clear from the proof above that the implication from right to left also holds for non-central extensions. In the general case, however, the equivalence fails. For example, when Q is a cyclic group and G non-abelian, we certainly have $\chi(N) \cap \mathrm{K}(G) = \mathrm{K}(G) > 1$, and the extension is CP.

4.1.3 CP subgroups

We proceed with some further characterizations of central CP extensions. We say that a normal abelian subgroup N of a group G is a *CP subgroup* of G if the extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$$

is a CP extension. In the case when N is central in G, Proposition 4.10 implies that N is a CP subgroup if and only $N \cap \mathrm{K}(G) = 1$. The following lemma will be needed.

Lemma 4.11. *Let N be a central CP subgroup of G. Then the sequences*

$$0 \longrightarrow \mathrm{B}_0(G) \longrightarrow \mathrm{B}_0(G/N) \longrightarrow N \cap G' \longrightarrow 0$$

and

$$N \otimes G^{\mathrm{ab}} \to \mathrm{M}_0(G) \to \mathrm{M}_0(G/N) \to 0$$

are exact.

Proof. Let G and N be given via free presentations, that is, $G = F/R$ and $N = S/R$. The fact that N is a central CP subgroup of G is then equivalent to $\langle \mathrm{K}(F) \cap S\rangle \leq R$. This immediately implies that $\langle \mathrm{K}(F) \cap S\rangle = \langle \mathrm{K}(F) \cap R\rangle$. With the above identifications and the Hopf-type formula for the Bogomolov multiplier, we have that $\mathrm{B}_0(G) = (F' \cap R)/\langle \mathrm{K}(F) \cap R\rangle$, $\mathrm{B}_0(G/N) = (F' \cap S)/\langle \mathrm{K}(F) \cap S\rangle$, $\mathrm{M}_0(G) = \langle \mathrm{K}(F) \cap R\rangle/[F, R]$, and $\mathrm{M}_0(G/N) = \langle \mathrm{K}(F) \cap S\rangle/[F, S]$. By [BT82, p. 41] there is a Ganea map $N \otimes G^{\mathrm{ab}} \to \mathrm{M}(G)$ whose image can be identified with $[F, S]/[F, R]$. As $[F, S] \leq \langle \mathrm{K}(F) \cap R\rangle$, the Ganea map actually maps $N \otimes G^{\mathrm{ab}}$ into $\mathrm{M}_0(G)$. The rest of the proof is now straightforward. $\square$

Proposition 4.12. *Let N be a central subgroup of a group G. The following are equivalent:*

1. *N is a CP subgroup of G.*
2. *The canonical map $\mathrm{M}_0(G) \to \mathrm{M}_0(G/N)$ is surjective.*
3. *The canonical map $\varphi : G \curlywedge G \to G/N \curlywedge G/N$ is an isomorphism.*

Proof. Let $G = F/R$ and $N = S/R$ be free presentations of G and N. Then the image of the map $\mathrm{M}_0(G) \to \mathrm{M}_0(G/N)$ can be identified with $\langle \mathrm{K}(F) \cap R\rangle/[F, S]$. Thus the above map is surjective if and only if $\langle \mathrm{K}(F) \cap R\rangle = \langle \mathrm{K}(F) \cap S\rangle$. In particular, $\langle \mathrm{K}(F) \cap S\rangle \leq R$, therefore N is a CP subgroup of G. This, together with Lemma 4.11, shows that the first and second claim are equivalent. Furthermore, from Proposition 3.9 it follows that $\ker \varphi = \langle x \curlywedge y \mid [x, y] \in N\rangle$. Hence φ is injective if and only if $\mathrm{K}(G) \cap N = 1$, hence the first and third claim are equivalent. $\square$

4.1.4 Isoclinism

We now discuss comparing different extensions. Let

$$e_1 : \quad 1 \longrightarrow N_1 \xrightarrow{\chi_1} G_1 \xrightarrow{\pi_1} Q_1 \longrightarrow 1$$

and

$$e_2 : \quad 1 \longrightarrow N_2 \xrightarrow{\chi_2} G_2 \xrightarrow{\pi_2} Q_2 \longrightarrow 1$$

be central extensions. Following [BT82], we say that e_1 and e_2 are *isoclinic* if there exist isomorphisms $\eta : Q_1 \to Q_2$ and $\xi : G_1' \to G_2'$ such that the diagram

$$\begin{tikzcd} Q_1 \times Q_1 \arrow[r, "c_1"] \arrow[d, "\eta\times\eta"] & G_1' \arrow[d, "\xi"] \\ Q_2 \times Q_2 \arrow[r, "c_2"] & G_2' \end{tikzcd}$$

commutes, where the maps c_i, $i = 1, 2$, are defined by the rules $c_i(\pi_i(x), \pi_i(y)) = [x, y]$. Note that these are well defined, since the extensions are central.

Proposition 4.13. *Let e_1 and e_2 be isoclinic central extensions. If e_1 is a CP extension, then so is e_2.*

Proof. We use the same notations as above. Choose a commuting pair (x_2, y_2) of elements of Q_2. Denote $x_2 = \eta(x_1)$ and $y_2 = \eta(y_1)$ where $x_1, y_1 \in Q_1$. Clearly $[x_1, y_1] = 1$. As e_1 is a CP central extension, we can choose commuting lifts $g_1, h_1 \in G_1$ of x_1 and y_1, respectively. We can write $x_2 = \pi_2(g_2)$ and $y_2 = \pi_2(h_2)$ for some $g_2, h_2 \in G_2$. By definition, $1 = \xi([g_1, h_1]) = [g_2, h_2]$, hence g_2 and h_2 are commuting lifts in G_2 of x_2 and y_2, respectively. $\square$

We now show how CP extensions up to isoclinism of a given group can be obtained from an action of its Bogomolov multiplier.

Theorem 4.14. *The isoclinism classes of central CP extensions with factor group isomorphic to Q correspond to the orbits of the action of $\operatorname{Aut} Q$ on the subgroups of $\mathrm{B}_0(Q)$ given by $(\varphi, U) \mapsto \mathrm{B}_0(\varphi)U$, where $\varphi \in \operatorname{Aut} Q$ and $U \leq \mathrm{B}_0(Q)$.*

Proof. Let

$$e: \quad 1 \longrightarrow N \xrightarrow{\chi} G \xrightarrow{\pi} Q \longrightarrow 1$$

be a central CP extension. As $\chi(N) \cap K(G) = 1$, it follows from [BT82] and Theorem 3.6 that we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{B}_0(G) \arrow[r, "\mathrm{B}_0(\pi)"] & \mathrm{B}_0(Q) \arrow[r, "\tilde{\theta}(e)"] & N \arrow[r, "\tau"] & G^{\mathrm{ab}} \arrow[r] & Q^{\mathrm{ab}} \arrow[r] & 0 \\
 & \mathrm{M}(G) \arrow[r, "\mathrm{M}(\pi)"] \arrow[u, two heads] & \mathrm{M}(Q) \arrow[r, "\theta_*(e)"] \arrow[u, two heads] & N \arrow[r, "\tau"] \arrow[u, equal] & G^{\mathrm{ab}} \arrow[r] \arrow[u, equal] & Q^{\mathrm{ab}} \arrow[r] \arrow[u, equal] & 0 \\
 & \mathrm{M}_0(G) \arrow[r] \arrow[u, hook] & \mathrm{M}_0(Q) \arrow[u, hook] & & & &
\end{tikzcd}$$

By the exactness we have that the image of $\chi\tilde{\theta}(e)$ is equal to $\chi(N) \cap G'$ which equals to the image of $\chi\theta_*(e)$. Since χ is injective, it follows that $\tilde{\theta}(e)$ and $\theta_*(e)$ have the same image. Furthermore, we claim that $\ker\tilde{\theta}(e) = \ker\theta_*(e)/\mathrm{M}_0(Q)$. To this end, consider free presentations $G = F/R$, $N = S/R$, and $Q = F/S$. Since the extension e is CP, it follows that $\langle \mathrm{K}(F) \cap S\rangle \leq R$. With the above identifications we have that $\ker\theta_*(e) = (F' \cap R)/[F, S]$ and $\ker\tilde{\theta}(e) = (F' \cap R)/\langle \mathrm{K}(F) \cap S\rangle$. As $\mathrm{M}_0(Q) = \langle \mathrm{K}(F) \cap S\rangle/[F, S]$, the equality follows.

Let now

$$e_i: \quad 1 \longrightarrow N_i \xrightarrow{\chi_i} G_i \xrightarrow{\pi_i} Q_i \longrightarrow 1\ , \ (i = 1, 2)$$

be central CP extensions, and let $\eta : Q_1 \to Q_2$ be an isomorphism of groups. By [BT82, Proposition III.2.3] we have that η induces isoclinism between e_1 and e_2 if and only if $\mathrm{M}(\eta)\ker\theta_*(e_1) = \ker\theta_*(e_2)$. By the above, this is equivalent to $\mathrm{B}_0(\eta)\ker\tilde{\theta}(e_1) = \ker\tilde{\theta}(e_2)$. The proof of [BT82, Proposition III.2.6] can now be suitably modified to obtain the result, we skip the details. $\square$

4.2 Covering groups

4.2.1 Maximal CP extensions

We study maximal central CP extensions of a given group in this section. Maximal here refers to the size of the kernel in a suitable representative extension under isoclinism. Recall that an extension

$$1 \longrightarrow N \xrightarrow{\chi} G \longrightarrow Q \longrightarrow 1$$

is termed to be *stem* whenever $\chi(N) \leq [G, G]$. The motivation comes from the following lemma.

Lemma 4.15. *Every central CP extension is isoclinic to a stem central CP extension.*

Proof. The argument follows along the lines of [BT82, Proposition III.2.6]. Let

$$e: \quad 1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a central CP extension. Put $U = \ker \tilde{\theta}(e)$, where $\tilde{\theta}(e)$ is the homomorphism $\mathrm{B}_0(Q) \to N$ from the 5-term exact sequence in Theorem 3.6. The subgroup U of $\mathrm{B}_0(Q)$ determines a central CP extension $\bar{e}$ of $\mathrm{B}_0(Q)/U$ by Q via Theorem 4.8 applied to the epimorphism $\mathrm{B}_0(Q) \to \mathrm{B}_0(Q)/U$. Thus $\tilde{\theta}(\bar{e})$ corresponds to the natural projection $\mathrm{B}_0(Q) \to \mathrm{B}_0(Q)/U$. Note that $\bar{e}$ is a stem central CP extension isoclinic to e, cf. the proof of Theorem 4.14. The kernel of the extension $\bar{e}$ is precisely $\mathrm{B}_0(Q)/U \cong \operatorname{im} \tilde{\theta}(e) \cong \ker(N \to G/[G, G]) = N \cap [G, G]$. □

Up to isoclinism of extensions, it therefore suffices to consider stem central CP extensions.

Given a group Q, any stem central CP extension of a group N by Q with $|N| = |\mathrm{B}_0(Q)|$ is called a *CP cover* of Q. The following theorem is of fundamental importance and justifies the terminology.

Theorem 4.16. *Let Q be a finite group given via a free presentation $Q = F/R$. Set*

$$H = \frac{F}{\langle \mathrm{K}(F) \cap R \rangle} \quad \text{and} \quad A = \frac{R}{\langle \mathrm{K}(F) \cap R \rangle}.$$

1. *A is a finitely generated central subgroup of H and its torsion subgroup is*

$$T(A) = \frac{[F, F] \cap R}{\langle K(F) \cap R \rangle} \cong \mathrm{B}_0(Q).$$

2. *Let C be a complement to $T(A)$ in A. Then H/C is a CP cover of Q.*

3. *Let G be a stem central CP extension of a group N by Q. Then G is a homomorphic image of H and in particular N is a homomorphic image of* $\mathrm{B}_0(Q)$.

4. *Let G be a CP cover of Q with kernel N. Then $N \cong \mathrm{B}_0(Q)$ and G is isomorphic to a quotient of H by a complement of $T(A)$ in A.*

5. *CP covers of Q are precisely the stem central CP extensions of Q of maximal order.*

6. *CP covers of Q are represented by the cocycles $\tilde{\varphi}^{-1}(1_{\mathrm{B}_0(Q)})$ in $\mathrm{H}^2(Q, \mathrm{B}_0(Q))$, where $\tilde{\varphi}$ is the mapping induced by the Universal Coefficients Theorem 4.8.*

Proof. We apply the arguments from [Hup67, Hauptsatz V.23.5] in combination with the Hopf formula for the Bogomolov multiplier. The group A is finitely generated because R is of finite index in F. We have $A \leq Z(H)$ and so $|H : Z(H)| \leq |H : A| = |G|$ is finite. Thus $[H, H]$ is also finite and therefore $[H, H] \cap A = ([F, F] \cap R)/\langle \mathrm{K}(F) \cap R\rangle$ is contained in $T(A)$. As the group $R[F, F]/[F, F] \cong R/([F, F] \cap R)$ is free abelian, the torsion $T(A)$ is contained in $([F, F] \cap R)/\langle \mathrm{K}(F) \cap R\rangle$. Thus $T(A) = ([F, F] \cap R)/\langle K(F) \cap R\rangle$. Next, select a complement C to $T(A)$ in A. Then $A/C \leq Z(H/C) \cap ([H, H]C/C)$. We also have $(H/C)/(A/C) \cong G$ and $A/C \cong T(A) \cong \mathrm{B}_0(G)$. Therefore H/C is indeed a CP cover of G. To see that all CP covers are obtained in this way, take one, say $B \to L \to G$, and lift the epimorphism $L \to G$ to an epimorphism $F \to L$. Since the extension $L \to G$ is central, we obtain an epimorphism $F/[R, F] \to L$. The restriction of this epimorphism to the subgroup $T(A)$ maps onto the kernel B. Since $|T(A)| = |\mathrm{B}_0(G)| = |B|$, it follows that $B \cong \mathrm{B}_0(G)$. We may thus understand the extension $B \to L \to G$ in terms of the universal central extension $A \to H \to G$. The remaining claims now follow easily from [BT82, Hup67]. □

Corollary 4.17. *The number of CP covers of a group Q is at most $|\operatorname{Ext}(Q^{\mathrm{ab}}, \mathrm{B}_0(Q))|$. In particular, perfect groups have a unique CP cover.*

***Example* 4.18.** *Let Q be a 4- or 12-cover of $PSL(3, 4)$. The group Q is a quasi-simple group and it is shown in [Kun10] that $\mathrm{B}_0(Q) \cong C_2$, so Q has a unique proper CP cover.*

We stress an important difference between Schur covering groups and CP covers, indicating a more intimate connection of the latter with the theory of (universal) covering spaces from algebraic topology [Hat02].

Theorem 4.19. *The Bogomolov multiplier of a CP cover is trivial.*

Proof. Let G be a CP cover of Q with kernel $N \cong \mathrm{B}_0(Q)$ satisfying $N \leq Z(G) \cap [G, G]$ and $N \cap \mathrm{K}(G) = 1$. Consider a CP cover $H \xrightarrow{\pi} G$ with kernel $M \cong \mathrm{B}_0(G)$ satisfying $M \leq Z(H) \cap [H, H]$ and $M \cap \mathrm{K}(H) = 1$. The group H is a central extension of $L = \pi^{-1}(N)$ by Q, since π preserves commutativity. Moreover, we have $L \leq \pi^{-1}([G, G]) = [H, H]$ since $M \leq [H, H]$, and $L \cap \mathrm{K}(H) \leq \pi^{-1}(N \cap \mathrm{K}(G)) \cap \mathrm{K}(H) \leq M \cap \mathrm{K}(H) = 1$. We conclude that H is a stem central CP extension of L by Q, therefore $|L| \leq |\mathrm{B}_0(Q)|$ by Theorem 4.16, and so $L \cong \mathrm{B}_0(Q)$. This implies $M = \mathrm{B}_0(G) = 1$, as required. □

Note that a similar proof gives that the Bogomolov multiplier of a Schur covering group is also trivial, see [FV91, Lemma 2.4.1].

For further use of Theorem 4.19, we record a straightforward corollary of Lemma 4.11.

Lemma 4.20. *Whenever N is a central CP subgroup of a group G with $\mathrm{B}_0(G) = 0$, then $\mathrm{B}_0(G/N) \cong N \cap [G,G]$. If in addition $N \leq [G,G]$, then the group G is a CP cover of G/N with kernel $N \cong \mathrm{B}_0(G/N)$.*

It follows readily that central CP extensions behave much as topological covering spaces.

Corollary 4.21. *Let Q be a group and G a CP cover of Q. For every filtration of subgroups*

$$1 = N_0 \leq N_1 \leq \cdots \leq N_n = \mathrm{B}_0(Q),$$

there is a corresponding sequence of groups $G_i = G/N_i$, where G_i is a central CP extension of G_j with kernel $N_j/N_i \cong \mathrm{B}_0(G_j)/\mathrm{B}_0(G_i)$ whenever $i \leq j$.

We now explore CP covers with respect to isoclinism. At first we list some auxiliary results.

Lemma 4.22. *Let*

$$e:\quad 1 \longrightarrow N \xrightarrow{\chi} G \xrightarrow{\pi} Q \longrightarrow 1$$

be a central CP extension. Then $\pi(Z(G)) = Z(Q)$ and $Z(G) \cong N \times Z(Q)$.

Proof. It is straightforward to see that if

$$e_i:\quad 1 \longrightarrow N \xrightarrow{\chi_i} G_i \xrightarrow{\pi_i} Q \longrightarrow 1$$

are equivalent central extensions for $i = 1, 2$, then $\pi_1(Z(G_1)) = \pi_2(Z(G_2))$. Thus we may replace the extension e by the extension

$$1 \longrightarrow N \longrightarrow G[\omega] \xrightarrow{\epsilon} Q \longrightarrow 1\ ,$$

that is obtained similarly as in the proof of Proposition 4.4. As $\omega \in \mathrm{Z}^2_{\mathrm{CP}}(Q,N)$, the condition that $(n,q) \in Z(G[\omega])$ is equivalent to $q \in Z(Q)$. Therefore $\epsilon(Z(G[\omega])) = Z(Q)$. $\square$

Lemma 4.23. *Let G be a CP cover of Q. Then $Z(G) \cong Z(Q) \times \mathrm{B}_0(Q)$, and G is stem if and only if Q is stem.*

Proof. The first part follows from Lemma 4.22. The second part then follows from the first one and the fact that $\mathrm{B}_0(Q) \leq [G,G]$. $\square$

It follows from the latter lemma that the central quotient of a CP cover is naturally isomorphic to the central quotient of the base group, and so the nilpotency class of a CP cover does not exceed that of the base group. This is all a special case of the following observation.

Proposition 4.24. *CP covers of isoclinic groups are isoclinic.*

Proof. Let G_1 be a CP cover of a group Q_1 with the covering projection $p_1\colon G_1 \to Q_1$ and let Q_2 be isoclinic to Q_1 via the compatible pair of isomorphisms $\alpha\colon Q_2/Z(Q_2) \to Q_1/Z(Q_1)$ and $\beta\colon [Q_2, Q_2] \to [Q_1, Q_1]$. Let G_2 be a CP cover of Q_2 with the covering projection $p_2\colon G_2 \to Q_2$. We show that G_2 is isoclinic to G_1. To this end, let $\bar{p}_i\colon G_i/Z(G_i) \to Q_i/Z(Q_i)$ be the natural homomorphisms induced by p_i's. Lemma 4.15 implies that $\bar{p}_i$ is in fact an isomorphism. Define $\tilde{\alpha}\colon G_2/Z(G_2) \to G_1/Z(G_1)$ as $\tilde{\alpha} = (\bar{p}_1)^{-1} \circ \alpha \circ \bar{p}_2$. This is clearly an isomorphism. Next, observe that Theorem 4.16 shows that the covering projections p_i also induce isomorphisms $p_i \curlywedge p_i\colon [G_i, G_i] \to Q_i \curlywedge Q_i$ defined as $[x, y] \mapsto p_i(x) \curlywedge p_i(y)$. Furthermore, it is shown in [Mor14] that α induces an isomorphism $\alpha^{\curlywedge}\colon Q_2 \curlywedge Q_2 \to Q_1 \curlywedge Q_1$ via $\alpha^{\curlywedge}(x_1 \curlywedge x_2) = y_1 \curlywedge y_2$, where $y_i Z(Q_1) = \alpha(x_i Z(Q_2))$. Now define $\tilde{\beta}\colon [G_2, G_2] \to [G_1, G_1]$ as $\tilde{\beta} = (p_1 \curlywedge p_1)^{-1} \circ \alpha^{\curlywedge} \circ (p_2 \curlywedge p_2)$. This is clearly an isomorphism, and it readily follows from the compatibility relations between α and β that the isomorphisms $\tilde{\alpha}$ and $\tilde{\beta}$ are also compatible. These induce an isoclinism between the CP covers G_1 and G_2. □

As a corollary, the derived subgroup of a CP cover is uniquely determined. Note that given a group Q and its CP cover G, we have $[G, G] \cong Q \curlywedge Q$ by Theorem 4.16. In particular, groups belonging to the same isoclinism family have naturally isomorphic curly exterior squares, and therefore also Bogomolov multipliers.

Let Φ be an isoclinism family of finite groups, referred to as the *base family*, and let G be an arbitrary group in Φ. By Proposition 4.24, CP covers of G all belong to the same isoclinism family. We denote this family by $\tilde{\Phi}$ and call it the *covering family* of Φ.

Proposition 4.25. *Every group in a covering family is a CP cover of a group in the base family.*

Proof. Let G_1 be a CP cover of a group Q_1 with the covering projection $p_1\colon G_1 \to Q_1$ and let G_2 be isoclinic to G_1 via the compatible pair of isomorphisms $\alpha\colon G_2/Z(G_2) \to G_1/Z(G_1)$ and $\beta\colon [G_2, G_2] \to [G_1, G_1]$. By Theorem 4.19, we have $\mathrm{B}_0(G_1) = 0$, and so $\mathrm{B}_0(G_2) = 0$ by Theorem 3.3. The commutator homomorphism $\kappa_i\colon G_i \curlywedge G_i \to [G_i, G_i]$ is therefore an isomorphism, and we implicitly identify the two groups. Consider the group $N = \beta^{-1}\,\mathrm{B}_0(Q_1) \leq [G_2, G_2]$. Note that N is central in G_2. Furthermore, whenever $[x_1, x_2] \in N$ for some $x_1, x_2 \in G_2$ with $\alpha(x_i Z(G_2)) = y_i Z(G_1)$, we have $[y_1, y_2] = \beta([x_1, x_2]) \in \mathrm{B}_0(Q_1)$, and so $[x_1, x_2] = \beta^{-1}([y_1, y_2]) = 1$ since the covering projection $G_1 \to Q_1$ is commutativity preserving. Now put $Q_2 = G_2/N$. By Lemma 4.20, the group G_2 is a CP cover of Q_2 with kernel $N \cong \mathrm{B}_0(Q_2)$. Finally, it is straightforward that the isomorphisms α and β naturally induce an isoclinism between the groups $Q_2 = G_2/\beta^{-1}(\mathrm{B}_0(Q_1))$ and $G_1/\mathrm{B}_0(Q_1) \cong Q_1$. □

Note that Lemma 4.23 now implies that CP covers of the stem of the base family form the stem of the covering family.

The following examples show that a given isoclinism family can be a covering family for more than one base family. Moreover, a group in a covering family can be a CP cover of non-isomorphic groups belonging to the same base family.

***Example* 4.26.** *Consider the isoclinism family that contains groups of smallest possible order having non-trivial Bogomolov multipliers [CHKK10]. This is the family Φ_{16} of [JNO90]. Its stem groups are of order* 64*, and its covering family $\tilde{\Phi}_{16}$ is the isoclinism family Φ_{36} of [JNO90], whose stem groups are of order* 128*.*

***Example* 4.27.** *Let G be a Schur covering group of the abelian group C_4^4, generated by g_1, g_2, g_3, g_4. Put $w = [g_1, g_2][g_3, g_4]$ and set $G_1 = G/\langle w\rangle$, $G_2 = G/\langle w^2\rangle$. It is readily verified that neither w nor w^2 is a commutator in G. Since $\mathrm{B}_0(G) = 0$, if follows that G is a CP cover of both G_1 and of G_2. Applying Theorem 3.6 gives $\mathrm{B}_0(G_1) \cong C_4$ and $\mathrm{B}_0(G_2) \cong C_2$, so G_1 and G_2 do not belong to the same isoclinism family.*

***Example* 4.28.** *Let Q be a stem group in the family Φ_{30} of [JNO90] and let G be a CP cover of Q. In the following section, we will show that $\mathrm{B}_0(Q) = \langle w_1\rangle \times \langle w_2\rangle \cong C_2 \times C_2$ for some $w_1, w_2 \in G$. Set $G_1 = G/\langle w_1\rangle$ and $G_2 = G/\langle w_2\rangle$. The groups G_1 and G_2 are isoclinic and non-isomorphic groups of order* 256*, and G is a CP cover of both of them. It can be verified that the groups G_1, G_2 in fact have exactly two non-isomorphic CP covers in common.*

It is well-known that Schur covering groups of a given group are all isoclinic, see for example [Hup67, Satz V.23.6]. Neither Proposition 4.24 nor Proposition 4.25, however, has a counterpart in the theory of Schur covering groups, as already the following simple example shows.

***Example* 4.29.** *Let Φ be the isoclinism family of all finite abelian groups. We plainly have $\tilde{\Phi} = \Phi$. Let p be an arbitrary prime. The Schur cover of C_{p^2} is C_{p^2}, and the Schur cover of $C_p \times C_p$ is isomorphic to the unitriangular group $UT_3(p)$. The two covers are not isoclinic. Note also that the group $C_p \times C_p$ is not a Schur covering group of any group.*

4.2.2 Minimal CP extensions

In this section, we focus on central CP extensions of a cyclic group of prime order by some given group Q. We call such extensions minimal CP extensions. By Corollary 4.21, every central CP extension is built from a sequence of such minimal extensions. As in the classical theory of central extensions, this corresponds to considering $\mathbb{F}_p$-cohomology. We thus set $\mathrm{H}^2_{\mathrm{CP}}(Q) = \mathrm{H}^2_{\mathrm{CP}}(Q, \mathbb{F}_p)$, the action of Q on $\mathbb{F}_p$ being trivial. Relying on Theorem 4.16, the heart of the matter here is relating a given presentation of Q with the object $\mathrm{H}^2_{\mathrm{CP}}(Q)$. The following result is obtained.

Theorem 4.30. *The group $\mathrm{H}^2_{\mathrm{CP}}(Q)$ is elementary abelian of rank $\mathrm{d}(Q) + \mathrm{d}(\mathrm{B}_0(Q))$.*

Proof. Let $Q = F/R$ be a presentation of Q. Consider first the canonical central CP extension $H = F/\langle \mathrm{K}(F) \cap R\rangle$ of Q. The kernel of this extension is the group $A = R/\langle \mathrm{K}(F) \cap R\rangle$.

We first claim that $\mathrm{H}^2_{\mathrm{CP}}(H) = 0$. By Theorem 3.6, we have $\mathrm{B}_0(H) = 0$, and it then follows from Theorem 4.8 that $\mathrm{H}^2_{\mathrm{CP}}(H) = \mathrm{Ext}(H^{\mathrm{ab}}, \mathbb{F}_p) = 0$.

Next we show that the minimal CP extensions are precisely the kernel of the inflation map from Q to H:

$$\mathrm{H}^2_{\mathrm{CP}}(Q) = \ker\left(\inf_Q^H\colon\ \mathrm{H}^2(Q) \to \mathrm{H}^2(H)\right).$$

Indeed, it follows from the above claim that $\mathrm{H}^2_{\mathrm{CP}}(Q) \leq \ker\inf_Q^H$. Conversely, let $\omega \in \ker\inf_Q^H$. Hence there is a function $\phi\colon H \to \mathbb{F}_p$ such that $\inf_Q^H(\omega)(x_1, x_2) = \phi(x_1) + \phi(x_2) - \phi(x_1x_2)$. Pick any commuting pair $u, v \in Q$. Then there exists a commuting lift $\tilde{u}, \tilde{v} \in H$ of these elements. Therefore $\omega(u, v) = \inf_Q^H(\omega)(\tilde{u}, \tilde{v}) = \inf_Q^H(\omega)(\tilde{v}, \tilde{u}) = \omega(v, u)$, and so $\omega \in \mathrm{H}^2_{\mathrm{CP}}(Q)$.

Let us now restrict to choosing the presentation $Q = F/R$ to be minimal in the sense that $\mathrm{d}(Q) = \mathrm{d}(F)$. In this case, we invoke the inflation-restriction cohomological exact sequence for the surjection $H \to Q$ with kernel A. Together with the above, it immediately follows that $\mathrm{H}^2_{\mathrm{CP}}(Q) \cong \mathrm{Hom}(A, \mathbb{F}_p)$. Finally, we have by Theorem 4.16 that the torsion $T(A) \cong \mathrm{B}_0(Q)$ in A has a free complement of rank $\mathrm{d}(F) = \mathrm{d}(Q)$. The proof is complete. □

We expose a corollary of the above proof.

Corollary 4.31. *Let $Q = F/R$ be a presentation with $\mathrm{d}(Q) = \mathrm{d}(F)$. Let $\mathrm{r}(F, R)$ be the minimal number of relators in R that generate R as a normal subgroup of F, and let $\mathrm{r}_{\mathrm{K}}(F, R)$ be the number of relators among these that belong to $\mathrm{K}(F)$. Then $\mathrm{d}(\mathrm{B}_0(Q)) \leq \mathrm{r}(F, R) - \mathrm{r}_{\mathrm{K}}(F, R) - \mathrm{d}(Q)$.*

Proof. Going back to the proof of Theorem 4.30, it is clear that $\operatorname{rank} A \leq \mathrm{r}(F, R) - \mathrm{r}_{\mathrm{K}}(F, R)$. The claim follows immediately. □

The corollary may be applied to show that the Bogomolov multiplier of a group is trivial. This works with classes of groups which may be given by a presentation with many simple commutators among relators. As an example, the group of unitriangular matrices $\mathrm{UT}_n(p)$ has a presentation in which all relators are commutators, whence immediately $\mathrm{B}_0(\mathrm{UT}_n(p)) = 0$. The same holds for lower central quotients of $\mathrm{UT}_n(p)$. Another example is the braid group B_n with $n-1$ generators and $n-2$ braid relators that are not commutators, thereby again $\mathrm{B}_0(B_n) = 0$. Finally, take G to be a p-group of maximal class with a free presentation as in [LGM02, Exercise 3.3(4)]. It follows that $\operatorname{rank} \mathrm{B}_0(G) \leq (p-1)/2$.

4.3 Computations

Using Theorem 4.16, a fast algorithm for computing the Schur covering groups of finite solvable groups as developed in [Nic93] may be adapted to give an algorithm for determining the CP cover and Bogomolov multiplier of a given group. We will focus here more on computing curly exterior squares and Bogomolov multipliers. The algorithm we give is able to recognize the commutator relations of the group that constitute its Bogomolov multiplier. As a sample case we will use the algorithm to effectively determine the multipliers of groups of order 128.

4.3.1 The algorithm

An algorithm for computing $\mathrm{B}_0(G)$ and $G \curlywedge G$ when G is a polycyclic group was first developed in [Mor12]. It is based on an algorithm for computing Schur multipliers that was developed by Eick and Nickel [EN08] and the Hopf-type formula for $\mathrm{B}_0(G)$. We will describe this algorithm, together with some additional refinements which make it more effective. An advantage of this new algorithm is that it enables a systematic trace of which elements of $\mathrm{B}_0(G)$ are in fact non-trivial, thus providing an efficient tool of double-checking non-triviality of Bogomolov multipliers by hand. The algorithm has been implemented in GAP [GAP] and is available at the website [JM14 GAP]. We remark that Ellis developed another algorithm for computing Bogomolov multipliers of arbitrary finite groups. It is available as a part of a homological algebra library HAP, cf. [HAP] for further details.

Let G be a finite polycyclic group, presented by a power-commutator presentation with a polycyclic generating sequence g_i with $1 \leq i \leq n$ for some n subject to the relations

$$\begin{aligned} g_i^{e_i} &= \textstyle\prod_{k=i+1}^n g_k^{x_{i,k}} && \text{for } 1 \leq i \leq n, \\ [g_i, g_j] &= \textstyle\prod_{k=i}^n g_k^{y_{i,j,k}} && \text{for } 1 \leq j < i \leq n, \end{aligned}$$

where $0 \leq x_{i,k}, y_{i,jk} < e_k$. Note that when printing such a presentation, we hold to standard practice and omit the trivial commutator relations, i.e. those for which $y_{i,j,k} = 0$ for all k. For every relation except the trivial commutator relations (the reason being these get factored out in the next step), introduce a new abstract generator, a so-called *tail*, append the tail to the relation, and make it central. In this way, we obtain a group generated by g_i with $1 \leq i \leq n$ and t_ℓ with $1 \leq \ell \leq m$ for some m, subject to the relations

$$\begin{aligned} g_i^{e_i} &= \textstyle\prod_{k=i+1}^n g_k^{x_{i,k}} \cdot t_{\ell(i)} && \text{for } 1 \leq i \leq n, \\ [g_i, g_j] &= \textstyle\prod_{k=i}^n g_k^{y_{i,j,k}} \cdot t_{\ell(i,j)} && \text{for } 1 \leq j < i \leq n, \end{aligned}$$

with the tails t_ℓ being central. This presentation gives a central extension $G^*_\emptyset$ of $\langle t_\ell \mid 1 \leq \ell \leq m\rangle$ by G, but the given relations may not determine a consistent power-commutator presentation. Evaluating the consistency relations

$$\begin{aligned} g_k(g_j g_i) &= (g_k g_j) g_i && \text{for } k > j > i, \\ (g_j^{e_j}) g_i &= g_j^{e_j - 1}(g_j g_i) && \text{for } j > i, \\ g_j(g_i^{e_i}) &= (g_j g_i) g_i^{e_i - 1} && \text{for } j > i, \\ (g_i^{e_i}) g_i &= g_i (g_i^{e_i}) && \text{for all } i \end{aligned}$$

in the extension gives a system of relations between the tails. Having these in mind, the above presentation of $G^*_\emptyset$ amounts to a pc-presented quotient of the universal central extension G^* of the quotient system in the sense of [Nic93], backed by the theory of the *tails routine* and *consistency checks*, see [Nic93, Sim94, EN08]. Beside the consistency enforced relations, we evaluate the commutators $[g, h]$ in the extension with the elements g, h commuting in G, which potentially impose some new tail relations. In the language of exterior squares, this step amounts to determining the subgroup $\mathrm{M}_0(G)$ of the Schur multiplier. This is computationally the most demanding part of the algorithm, since it does in general not suffice to inspect only commuting pairs made up of the polycyclic generators. The procedure may be simplified by noticing that the conjugacy class of a single commutator induces the same relation throughout. For this purpose, we work with a pc-presented version of the group in our algorithm, for which the implemented algorithm for determining conjugacy classes in GAP is much faster than the corresponding one for polycyclic groups. Let G_0^* be the group obtained by factoring $G_\emptyset^*$ by these additional relations. Computationally, we do this by applying Gaussian elimination over the integers to produce a generating set for all of the relations between the tails at once, and collect them in a matrix T. Applying a transition matrix Q^{-1} to obtain the Smith normal form of $T = PSQ$ gives a new basis for the tails, say t_ℓ^*. The abelian invariants of the group generated by the tails are recognized as the elementary divisors of T. Finally, the Bogomolov multiplier of G is identified as the torsion subgroup of $\langle t_\ell^* \mid 1 \leq \ell \leq m\rangle$ inside G_0^*, the theoretical background of this being the following restatement of Theorem 4.16 in the present context.

Proposition 4.32. *Let G be a finite group, presented by $G = F/R$ with F free of rank n. Denote by $\mathrm{K}(F)$ the set of commutators in F. Then $\mathrm{B}_0(G)$ is isomorphic to the torsion subgroup of $R/\langle \mathrm{K}(F) \cap R\rangle$, and the torsion-free factor $R/([F,F] \cap R)$ is free abelian of rank n. Moreover, every complement C to $\mathrm{B}_0(G)$ in $R/\langle \mathrm{K}(F) \cap R\rangle$ yields a commutativity preserving central extension of $\mathrm{B}_0(G)$ by G.*

Proof. Everything follows from Theorem 4.16. By construction and [EN08], we have $G_0^* \cong F/\langle \mathrm{K}(F) \cap R\rangle$, and the complement C gives the extension G_0^*/C. □

Taking the derived subgroup of the extension G_0^* and factoring it by a complement of the torsion part of the subgroup generated by the tails thus gives a consistent power-commutator presentation of the curly exterior square $G \curlywedge G$, see [EN08, Mor12]. With each of the groups below, we also output the presentation of G_0^* factored by a complement of $\mathrm{B}_0(G)$ and expressed in the new tail basis t_i^* as to explicitly point to the nonuniversal commutator relations with respect to the commutator presentation of the original group.

Lastly, we compare our algorithm to the one given in [Mor12] and existing algorithms based on other approaches [HAP]. The original algorithm from [Mor12] was designed only to determine $\mathrm{B}_0(G)$; our approach furthermore explicitly constructs a central extension of the Bogomolov multiplier by the group G, which makes it possible to trace and in the end also recognize the commutator relations that constitute $\mathrm{B}_0(G)$. Moreover, our implementation adapts the algorithm [EN08] rather than directly extending it by not adding the tails that correspond to trivial commutators of the polycyclic generating sequence in the first place. With respect to more homological, cohomological and tensor implementations [HAP], our algorithm is specialized for polycyclic groups. As such, it is as a rule more efficient, particularly with groups of larger orders. This is of course also a limitation of our algorithm, but in fact not a big obstacle, since the p-part of $\mathrm{B}_0(G)$ embeds into $\mathrm{B}_0(S)$, where S is the Sylow p-subgroup of G.

Time tests on different classes of groups are presented in Table 4.1, time is given in seconds. These have been run on a standard laptop computer.

Table 4.1: Time tests of the algorithm.

`SmallGroup(128,100)`	0.08
`SmallGroup(128,1544)`	0.07
`AllSmallGroups(128)`	207.43
`DihedralGroup(2**14)`	25.52
`UnitriangularGroup(5,3)`	1.25

4.3.2 Groups of order 128

Hand calculations of Bogomolov multipliers were done for groups of order 32 by Chu, Hu, Kang, and Prokhorov [CHKP08], and groups of order 64 by Chu, Hu, Kang, and Kunyavskiĭ [CHKK10]. In a similar way, Bogomolov multipliers of groups of order p^5 were determined in [HK11, HKK12], and for groups of order p^6 this was done recently by Chen and Ma [CM13]. We apply the above algorithm to determine Bogomolov multipliers of all groups of order 128 by providing an explicit description of the generators of Bogomolov multipliers of these groups. There are 2328 groups of order 128, and they were classified by James, Newman, and O'Brien [JNO90]. Instead of considering all of them, we use the fact that these groups belong to 115 isoclinism families. It turns out that there are precisely eleven isoclinism families whose Bogomolov multipliers are non-trivial. For each of these families we explicitly determine $\mathrm{B}_0(G)$ for a chosen representative G. Their multipliers are all isomorphic to C_2, except those of the family Φ_{30} for which we get $C_2 \times C_2$. For each of the families with nontrivial multipliers, we also give the identification number as implemented in GAP of a selected representative that was used for determining the family's multiplier. The results are collected in Table 4.3. An extended note where the calculations for all 115 isoclinism families are described is posted at [JM13 arxiv]. All-in-all, there are 230 groups of order 128 with nontrivial Bogomolov multipliers out of a total of 2328 groups of this order.

16. Let the group G be the representative of this family given by the presentation

$$\begin{aligned}
\langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \mid \quad & g_1^2 = g_5, \\
& g_2^2 = 1, \quad [g_2, g_1] = g_4, \\
& g_3^2 = 1, \quad [g_3, g_1] = g_7, \quad [g_3, g_2] = g_6 g_7, \\
& g_4^2 = g_6, \quad [g_4, g_1] = g_6, \quad [g_4, g_2] = g_6, \\
& g_5^2 = g_7, \\
& g_6^2 = 1, \\
& g_7^2 = 1 \rangle.
\end{aligned}$$

Table 4.3: Isoclinism families of groups of order 128 with nontrivial Bogomolov multipliers.

Family	GAP ID	B_0
16	227	C_2
30	1544	$C_2 \times C_2$
31	1345	C_2
37	242	C_2
39	36	C_2
43	1924	C_2
58	417	C_2
60	446	C_2
80	950	C_2
106	144	C_2
114	138	C_2

We add 12 tails to the presentation as to form a quotient of the universal central extension of the system: $g_1^2 = g_5t_1$, $g_2^2 = t_2$, $[g_2, g_1] = g_4t_3$, $g_3^2 = t_4$, $[g_3, g_1] = g_7t_5$, $[g_3, g_2] = g_6g_7t_6$, $g_4^2 = g_6t_7$, $[g_4, g_1] = g_6t_8$, $[g_4, g_2] = g_6t_9$, $g_5^2 = g_7t_{10}$, $g_6^2 = t_{11}$, $g_7^2 = t_{12}$. Carrying out consistency checks gives the following relations between the tails:

$$\begin{aligned}
g_4^2g_2 = g_4(g_4g_2) &\implies & t_9^2t_{11} &= 1\\
g_4^2g_1 = g_4(g_4g_1) &\implies & t_8^2t_{11} &= 1\\
g_3^2g_2 = g_3(g_3g_2) &\implies & t_6^2t_{11}t_{12} &= 1\\
g_3^2g_1 = g_3(g_3g_1) &\implies & t_5^2t_{12} &= 1\\
g_2^2g_1 = g_2(g_2g_1) &\implies & t_3^2t_7t_9t_{11} &= 1\\
g_2g_1^2 = (g_2g_1)g_1 &\implies & t_3^2t_7t_8t_{11} &= 1
\end{aligned}$$

Scanning through the conjugacy class representatives of G and the generators of their centralizers, we see that no new relations are imposed. Collecting the coefficients of these relations into a matrix yields

$$T = \begin{array}{c}
\begin{matrix} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 & t_9 & t_{10} & t_{11} & t_{12} \end{matrix} \\
\begin{pmatrix}
 & & 2 & & & & 1 & & 1 & & 1 & \\
 & & & & 2 & & & & & & & 1 \\
 & & & & & 2 & & & & & 1 & 1 \\
 & & & & & & & 1 & 1 & & 1 & \\
 & & & & & & & & 2 & & 1 &
\end{pmatrix}
\end{array}.$$

A change of basis according to the transition matrix (specifying expansions of t_i^* by t_j)

$$
\begin{array}{c}
\\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \\ t_9 \\ t_{10} \\ t_{11} \\ t_{12}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
t_1^* & t_2^* & t_3^* & t_4^* & t_5^* & t_6^* & t_7^* & t_8^* & t_9^* & t_{10}^* & t_{11}^* & t_{12}^*
\end{array}\\
\begin{pmatrix}
 & & & & & -1 & -1 & 1 & & & 1 & \\
 & & & & & -1 & & -1 & & & -1 & -1 \\
-2 & & & & -2 & & & -1 & & & -1 & \\
 & & & & & 4 & & -1 & & & & \\
4 & & & & 3 & 1 & & & & & & \\
-16 & 2 & -2 & & -13 & -4 & & & & & & \\
-1 & & & & -1 & & 1 & & & & & \\
16 & -2 & 2 & 1 & 13 & & & 1 & & & & \\
-27 & 4 & -2 & -3 & -21 & & & & 1 & & & \\
 & & & & & & & & & 1 & & \\
-14 & 2 & -1 & -1 & -11 & & & & & & 1 & \\
-6 & 1 & -1 & & -5 & & & & & & & 1
\end{pmatrix}
\end{array}
$$

shows that the nontrivial elementary divisors of the Smith normal form of T are 1, 1, 1, 1, 2. The element corresponding to the divisor that is greater than 1 is t_5^*. This already gives $\mathrm{B}_0(G) \cong \langle t_5^* \mid t_5^{*2} \rangle$.

We now deal with explicitly identifying the nonuniversal commutator relation generating $\mathrm{B}_0(G)$. First, factor out by the tails t_i^* whose corresponding elementary divisors are either trivial or 1. Transforming the situation back to the original tails t_i, this amounts to the nontrivial expansion $t_6 = t_5^*$ and all the other tails t_i are trivial. We thus obtain a commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_5^*, subject to the following relations:

$$g_1^2 = g_5,\ g_2^2 = g_3^2 = 1,\ g_4^2 = g_6,\ g_5^2 = g_7,\ g_6^2 = g_7^2 = t_5^{*2} = 1,$$
$$[g_2, g_1] = g_4,\ [g_3, g_1] = g_7,\ [g_3, g_2] = g_6 g_7 t_5^*,\ [g_4, g_1] = g_6,\ [g_4, g_2] = g_6.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_5^* = [g_3, g_1][g_3, g_2]^{-1}[g_4, g_2]$.

30. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_4^*, t_5^*, subject to the following relations:

$$g_1^2 = g_2^2 = 1,\ g_3^2 = t_4^*,\ g_4^2 = g_5^2 = g_6^2 = g_7^2 = t_4^{*2} = t_5^{*2} = 1,$$
$$[g_2, g_1] = g_5,\ [g_3, g_1] = g_6 t_4^*,\ [g_3, g_2] = g_7 t_5^*,\ [g_4, g_2] = g_5 g_6,\ [g_4, g_3] = g_5 t_5^*.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relations of G are identified as $t_4^* = [g_2, g_1][g_3, g_1][g_4, g_2]^{-1}$ and $t_5^* = [g_2, g_1][g_4, g_3]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_4^*, t_5^* \mid t_4^{*2}, t_5^{*2} \rangle$.

31. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated

by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_4^*, subject to the following relations:

$$g_1^2 = g_2^2 = g_3^2 = g_4^2 = g_5^2 = g_6^2 = g_7^2 = t_4^{*2} = 1,$$
$$[g_2, g_1] = g_5,\ [g_3, g_1] = g_6t_4^*, [g_3, g_2] = g_7,\ [g_4, g_3] = g_5t_4^*.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_4^* = [g_2, g_1][g_4, g_3]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_4^* \mid t_4^{*2}\rangle$.

37. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_5^*, subject to the following relations:

$$g_1^2 = g_5t_5^*,\ g_2^2 = g_3^2 = 1,\ g_4^2 = g_7,\ g_5^2 = g_6^2 = g_7^2 = t_5^{*2} = 1,$$
$$[g_2, g_1] = g_4t_5^*,\ [g_3, g_1] = g_7t_5^*,\ [g_4, g_1] = g_6,\ [g_4, g_2] = g_7,\ [g_5, g_2] = g_6g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_5^* = [g_3, g_1][g_4, g_2]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_5^* \mid t_5^{*2}\rangle$.

39. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_5^*, subject to the following relations:

$$g_1^2 = g_4,\ g_2^2 = g_5,\ g_3^2 = t_5^*,\ g_4^2 = g_5^2 = g_6^2 = g_7^2 = t_5^{*2} = 1,$$
$$[g_2, g_1] = g_3,\ [g_3, g_1] = g_6t_5^*,\ [g_3, g_2] = g_7t_5^*,\ [g_4, g_2] = g_6,\ [g_5, g_1] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_5^* = [g_3, g_2][g_5, g_1]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_5^* \mid t_5^{*2}\rangle$.

43. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_6^*, subject to the following relations:

$$g_1^2 = t_6^*,\ g_2^2 = t_6^*,\ g_3^2 = 1,\ g_4^2 = t_6^*,\ g_5^2 = 1,\ g_6^2 = g_7,\ g_7^2 = t_6^{*2} = 1,$$
$$[g_2, g_1] = g_5,\ [g_3, g_1] = g_6t_6^*,\ [g_3, g_2] = g_5g_7t_6^*,\ [g_4, g_1] = g_5,\ [g_6, g_1] = g_7,\ [g_6, g_3] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_6^* = [g_3, g_2][g_4, g_1]^{-1}[g_6, g_3]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_6^* \mid t_6^{*2}\rangle$.

58. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_6^*, subject to the following relations:

$$g_1^2 = 1,\ g_2^2 = g_4,\ g_3^2 = 1,\ g_4^2 = g_6,\ g_5^2 = g_7,\ g_6^2 = g_7^2 = t_6^{*2} = 1,$$
$$[g_2, g_1] = g_4,\ [g_3, g_1] = g_5,\ [g_3, g_2] = g_6t_6^*,\ [g_4, g_1] = g_6,\ [g_5, g_1] = g_7,\ [g_5, g_3] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_6^* = [g_3, g_2][g_4, g_1]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_6^* \mid t_6^{*2} \rangle$.

60. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_5^*, subject to the following relations:

$$g_1^2 = t_5^*,\ g_2^2 = g_4,\ g_3^2 = g_5,\ g_4^2 = g_6,\ g_5^2 = g_7,\ g_6^2 = g_7^2 = t_5^{*2} = 1,$$
$$[g_2, g_1] = g_4 t_5^*,\ [g_3, g_1] = g_5 t_5^*,\ [g_3, g_2] = g_6 t_5^*,\ [g_4, g_1] = g_6,\ [g_5, g_1] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_5^* = [g_3, g_2][g_4, g_1]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_5^* \mid t_5^{*2} \rangle$.

80. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_5^*, subject to the following relations:

$$g_1^2 = t_5^*,\ g_2^2 = g_4 g_6,\ g_3^2 = 1,\ g_4^2 = g_6 g_7 t_5^*,\ g_5^2 = 1,\ g_6^2 = g_7,\ g_7^2 = t_5^{*2} = 1,$$
$$[g_2, g_1] = g_4 t_5^*,\ [g_3, g_1] = g_5 t_5^*,\ [g_3, g_2] = g_7 t_5^*,\ [g_4, g_1] = g_6 t_5^*,\ [g_6, g_1] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_5^* = [g_3, g_2][g_6, g_1]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_5^* \mid t_5^{*2} \rangle$.

106. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_9^*, subject to the following relations:

$$g_1^2 = g_4,\ g_2^2 = g_6 t_9^*,\ g_3^2 = g_6 g_7 t_9^*,\ g_4^2 = 1,\ g_5^2 = g_7,\ g_6^2 = g_7^2 = t_9^{*2} = 1,$$
$$[g_2, g_1] = g_3,\ [g_3, g_1] = g_5 t_9^*,\ [g_3, g_2] = g_6 t_9^*,\ [g_4, g_2] = g_5 g_6,\ [g_4, g_3] = g_6 g_7,$$
$$[g_5, g_1] = g_6,\ [g_5, g_2] = g_7,\ [g_5, g_4] = g_7,\ [g_6, g_1] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_9^* = [g_3, g_2][g_5, g_1]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_9^* \mid t_9^{*2} \rangle$.

114. Choosing a representative group G of this family and applying the algorithm, we obtain the commutativity preserving central extension of the tails subgroup by G, generated by the sequence g_1, g_2, g_3, g_4, g_5, g_6, g_7, t_9^*, subject to the following relations:

$$g_1^2 = g_4,\ g_2^2 = t_9^*,\ g_3^2 = g_6 t_9^*,\ g_4^2 = 1,\ g_5^2 = g_7,\ g_6^2 = g_7^2 = t_9^{*2} = 1,$$
$$[g_2, g_1] = g_3,\ [g_3, g_1] = g_5 t_9^*,\ [g_3, g_2] = g_6 t_9^*,\ [g_4, g_2] = g_5 g_6 g_7,\ [g_4, g_3] = g_6 g_7,$$
$$[g_5, g_1] = g_6,\ [g_5, g_2] = g_7,\ [g_5, g_4] = g_7,\ [g_6, g_1] = g_7.$$

Its derived subgroup is isomorphic to the curly exterior square $G \curlywedge G$, whence the nonuniversal commutator relation of G is identified as $t_9^* = [g_3, g_2][g_5, g_1]^{-1}$, and we have $\mathrm{B}_0(G) \cong \langle t_9^* \mid t_9^{*2} \rangle$.

4.4 Minimal relations

4.4.1 B_0-minimal groups

In this section, we deal with groups that are minimal with respect to possessing a nonuniversal commutator relation. More specifically, a finite group G is termed to be a B_0-*minimal group* whenever $\mathrm{B}_0(G) \neq 0$ and for every proper subgroup H of G and every proper normal subgroup N of G, we have $\mathrm{B}_0(H) = \mathrm{B}_0(G/N) = 0$. If G is a B_0-minimal group, the nontrivial elements of $\mathrm{B}_0(G)$ are called *minimal commutator relations* of G. These groups may be thought of as the building blocks of groups with nontrivial Bogomolov multipliers. The class of B_0-minimal groups is a subclass of the class of absolute γ-minimal factors defined by Bogomolov [Bog87]. A part of the theory we develop has already been investigated by Bogomolov using cohomological methods; the alternative approach we take via the exterior square provides new proofs and refines that work.

***Example* 4.33.** *Let G be the group*

$$\left\langle\, a, b, c \;\middle|\; \begin{array}{l} a^2 = b^2 = 1,\ c^2 = [a, c], \\ [c, b] = [c, a, a],\ [b, a] \ \textit{central, class } 3 \end{array} \right\rangle.$$

Another way of presenting G is by a polycyclic generating sequence g_i with $1 \leq i \leq 6$, subject to the following relations: $g_1^2 = g_2^2 = 1$, $g_3^2 = g_4 g_5$, $g_4^2 = g_5$, $g_5^2 = g_6^2 = 1$, $[g_2, g_1] = g_6$, $[g_3, g_1] = g_4$, $[g_3, g_2] = g_5$, $[g_4, g_1] = g_5$, and $[g_i, g_j] = 1$ for other $i > j$. This is one of the stem groups of the family Γ_{16} of [HS64]. Application of the algorithm from the previous section shows that $\mathrm{B}_0(G)$ is generated by the element $(g_3 \curlywedge g_2)(g_4 \curlywedge g_1)$ of order 2 in $G \curlywedge G$. The group G is one of the groups of the smallest order that have a nontrivial Bogomolov multiplier [CHKP08, CHKK10], so it is also of minimal order amongst all B_0-minimal groups.

The class of B_0-minimal groups can be studied in terms of isoclinism. An isoclinism family that contains at least one B_0-minimal group is called a B_0-*minimal family.* Note that not every group in a B_0-minimal family is itself B_0-minimal. For example, one may take a B_0-minimal group G, a nontrivial abelian group A, and form their direct product $G \times A \simeq G$. This group is clearly not B_0 minimal. We show, however, that the stem groups of B_0-minimal families are themselves B_0-minimal.

Proposition 4.34. *In a B_0-minimal family, every group possesses a B_0-minimal subsection. In particular, the stem groups in the family are all B_0-minimal.*

Proof. Let G be a B_0-minimal member of the given isoclinism family and $H \simeq G$ a group that is not a B_0-minimal group. Since $\mathrm{B}_0(H) \cong \mathrm{B}_0(G) \neq 0$, the group H has either a subgroup or a quotient, say K, with a nontrivial Bogomolov multiplier. By [Hal40], the subgroups and quotients of H belong to the same isoclinism families as the subgroups and quotients of G. It follows from B_0-minimality of G that the group K must be isoclinic to H. As $|K| < |H|$, repeating the process with K instead of H yields a subsection S of H that is B_0-minimal and isoclinic to H. In particular, the stem groups in a B_0-minimal family must be B_0-minimal, since they are groups of minimal order in the family. $\square$

Note also that not all B_0-minimal groups in a given family need be stem, as the following example shows.

***Example* 4.35.** *Let G be the group generated by elements g_i with $1 \leq i \leq 8$, subject to the following relations: $g_1^2 = g_5$, $g_2^2 = g_3^2 = 1$, $g_4^2 = g_6$, $g_5^2 = g_7$, $g_6^2 = 1$, $g_7^2 = g_8$, $g_8^2 = 1$, $[g_2, g_1] = g_4$, $[g_3, g_1] = g_8$, $[g_3, g_2] = g_6 g_8$, $[g_4, g_1] = g_6$, $[g_4, g_2] = g_6$, and $[g_i, g_j] = 1$ for other $i > j$. Using the algorithm, we see that G is a B_0-minimal group. Its Bogomolov multiplier is generated by the element $(g_3 \curlywedge g_2)(g_4 \curlywedge g_2)(g_3 \curlywedge g_1)$ of order 2 in $G \curlywedge G$. Since g_7 belongs to the center $Z(G)$ but not to the derived subgroup $[G, G]$, the group G is not a stem group. In fact, G is isoclinic to the group given in Example 4.33, both the isoclinism isomorphisms stemming from interchanging the generators g_2 and g_3.*

Applying standard homological arguments, we quickly observe that B_0-minimal groups are p-groups.

Proposition 4.36. *A B_0-minimal group is a p-group.*

Proof. Let G be a B_0-minimal group. Suppose p is a prime dividing the order of G. By Theorem 3.11, the p-part of $\mathrm{B}_0(G)$ embeds into $\mathrm{B}_0(S)$, where S is a Sylow p-subgroup of G. It thus follows from B_0-minimality that G is a p-group. $\square$

Hence B_0-minimal families are determined by their stem p-groups. Making use of recent results on Bogomolov multipliers of p-groups of small orders [HK11, HKK12, CHKK10, CHKP08, CM13], we determine the B_0-minimal families of rank at most 6 for odd primes p, and those of rank at most 7 for $p = 2$. In stating the proposition, the classifications [Jam80, JNO90] are used.

Proposition 4.37. *The B_0-minimal isoclinism families of p-groups with p an odd prime and of rank at most 6 are precisely the families Φ_i with $i \in \{10, 18, 20, 21, 36\}$ of [Jam80]. The B_0-minimal isoclinism families of 2-groups of rank at most 7 are precisely the families Φ_i with $i \in \{16, 30, 31, 37, 39, 80\}$ of [JNO90].*

Proof. Suppose first that p is odd. If the rank of the family is at most 4, we have $\mathrm{B}_0(G) = 0$ by [Bog87]. Next, if the rank equals 5, stem groups of the family have nontrivial Bogomolov multipliers if and only if they belong to the family Φ_{10} by [HK11, HKK12, Mor12 p5]. Further, if the rank is 6, then it follows from [CM13] that stem groups of the family have nontrivial Bogomolov multipliers if and only if they belong to one of the isoclinism families Φ_i with $i \in \{18, 20, 21, 36, 38, 39\}$. Note that the families Φ_{38} and Φ_{39} only exist when $p > 3$. The groups in the families Φ_{18}, Φ_{20} and Φ_{21} are of nilpotency class at most 3, so none of their proper quotients and subgroups can belong to the isoclinism family Φ_{10}. Hence these families are indeed B_0-minimal. Central quotients of stem groups in the families Φ_{38} and Φ_{39} belong to the family Φ_{10}, so these groups are not B_0-minimal. On the other hand, the center of the stem groups of the family Φ_{36} is of order p and the central quotients of these groups belong to the family Φ_9, so this family is B_0-minimal.

Now let $p = 2$. It is shown in [CHKP08, CHKK10] that the groups of minimal order having nontrivial Bogomolov multipliers are exactly the groups forming the stem of the isoclinism family Γ_{16} of [HS64], so this family is B_0-minimal. In the notation of [JNO90], it corresponds to Φ_{16}. Now consider the isoclinism families of rank 7. Their Bogomolov multipliers have been determined in the previous section. The families whose multipliers are nontrivial are precisely the families Φ_i with $i \in \{30, 31, 37, 39, 43, 58, 60, 80, 106, 114\}$. It remains to filter out the B_0-minimal families from this list. Making use of the presentations of representative groups of these families as given above, it is straightforward that stem groups of the families Φ_{43}, Φ_{106} and Φ_{114} contain a maximal subgroup belonging to the family Φ_{16}, which implies that these families are not B_0-minimal. Similarly, stem groups of the families Φ_{58} and Φ_{60} possess maximal quotient groups belonging to Φ_{16}, so these families are also not B_0-minimal. On the other hand, it is readily verified that stem groups of the families Φ_i with $i \in \{30, 31, 37, 39, 80\}$ have no maximal subgroups or quotients belonging to the family Φ_{16}, implying that these families are B_0-minimal. $\square$

4.4.2 Structure of B_0-minimal groups

We now turn our attention to the structure of general B_0-minimal groups. The upcoming lemma is of key importance in our approach.

Lemma 4.38. *Let G be a B_0-minimal p-group and $z = \prod_{i\in I}[x_i, y_i]$ a central element of order p in G. Then there exist elements $a, b \in G$ satisfying*

$$G = \langle a, b, x_i, y_i \;;\; i \in I\rangle, \quad [a, b] = z, \quad a \curlywedge b \neq \textstyle\prod_{i\in I}(x_i \curlywedge y_i).$$

Proof. Let w be a nontrivial element of $\mathrm{B}_0(G)$ and put $N = \langle z\rangle$. The canonical projection $G \to G/N$ induces a homomorphism $G \curlywedge G \to G/N \curlywedge G/N \cong (G \curlywedge G)/J$, where $J = \langle a \curlywedge b \mid [a, b] \in N\rangle$ by Proposition 3.9. By B_0-minimality of G, the element w is in the kernel of this homomorphism, so it must belong to J. Suppose first that J is cyclic. Then there exist elements $x, y \in G$ with $[x, y] = z$ and $J = \langle x \curlywedge y\rangle$. Since $w \in J$, we have $w = (x \curlywedge y)^n$ for some integer n. Applying the commutator mapping, we obtain $1 = [x, y]^n = z^n$, so n must be divisible by p. But then $w = (x \curlywedge y)^n = x^n \curlywedge y = 1$, since z is central in G. This shows that J cannot be cyclic. Hence there exist elements $\tilde{a}, b \in G$ with $\prod_{i\in I}(x_i \curlywedge y_i) \notin \langle \tilde{a} \curlywedge b\rangle$ and $1 \neq [\tilde{a}, b] \in N$. The latter implies $[\tilde{a}, b] = z^m$ for some integer m coprime to p. Let μ be the multiplicative inverse of m modulo p and put $a = \tilde{a}^\mu$. The product $\prod_{i\in I}(x_i \curlywedge y_i)(a \curlywedge b)^{-1}$ is then a nontrivial element of $\mathrm{B}_0(G)$, since $[a, b] = z^{m\mu} = z$. By B_0-minimality of G, the subgroup generated by a, b, x_i, y_i, $i \in I$, must equal the whole of G. $\square$

The above proof immediately implies the following result which can be compared with [Bog87, Theorem 4.6].

Corollary 4.39. *The Bogomolov multiplier of a* B_0*-minimal group is of prime exponent.*

Proof. Let G be a B_0-minimal p-group and w a nontrivial element of $\mathrm{B}_0(G)$. For any central element z in G of order p, we have $w \in J_z = \langle a \curlywedge b \mid [a,b] \in \langle z \rangle \rangle$ by B_0-minimality, thus $w^p = 1$, as required. $\square$

We apply Lemma 4.38 to some special central elements of prime order in a B_0-minimal group. In this way, some severe restrictions on the structure of B_0-minimal groups are obtained. Recall that the *Frattini rank* of a group G is the cardinality of the smallest generating set of G.

Theorem 4.40. *A* B_0*-minimal group has an abelian Frattini subgroup and is of Frattini rank at most* 4*. Moreover, when the group is of nilpotency class at least* 3*, it is of Frattini rank at most* 3*.*

Proof. Let G be a B_0-minimal group and $\Phi(G)$ its Frattini subgroup. Suppose that $\Phi(G)$ is not abelian. Since G is a p-group, we have $[\Phi(G), \Phi(G)] \cap Z(G) \neq 1$, so there exists a central element z of order p in $[\Phi(G), \Phi(G)]$. Expand it as $z = \prod_i [x_i, y_i]$ with $x_i, y_i \in \Phi(G)$. By Lemma 4.38, there exist $a, b \in G$ so that the group G may be generated by the elements a, b, x_i, y_i, $i \in I$. Since the generators x_i, y_i belong to $\Phi(G)$, they may be omitted, and so $G = \langle a, b \rangle$. As the commutator $[a,b]$ is central in G, we have $[G,G] = \langle [a,b] \rangle \cong \mathbb{Z}/p\mathbb{Z}$. It follows from here that the exponent of $G/Z(G)$ equals p, so we finally have $\Phi(G) = G^p[G,G] \leq Z(G)$, a contradiction. This shows that the Frattini subgroup of G is indeed abelian. To show that the group G is of Frattini rank at most 4, pick any $x \in \gamma_{c-1}(G)$ and let $z = [x,y] \in \gamma_c(G)$ be an element of order p in G. By Lemma 4.38, there exist $a, b \in G$ so that the group G may be generated by a, b, x, y. Hence G is of Frattini rank at most 4. When the nilpotency class of G is at least 3, we have $x \in \gamma_{c-1}(G) \leq [G,G]$, so the element x is a nongenerator of G. This implies that G is of Frattini rank at most 3 in this case. $\square$

Note that in particular, Theorem 4.40 implies that a B_0-minimal group is metabelian, as was already shown in [Bog87, Theorem 4.6].

Corollary 4.41. *The exponent of the center of a stem* B_0*-minimal group divides* p^2*.*

Proof. Let G be a stem B_0-minimal group. Then $Z(G) \leq [G,G]$, and it follows from [Mor12, Proposition 3.12] that $Z(G)$ may be generated by central commutators. For any $x, y \in G$ with $[x,y] \in Z(G)$, we have $[x^p, y^p] = 1$ by Theorem 4.40, which reduces to $[x,y]^{p^2} = 1$ as the commutator $[x,y]$ is central in G. This completes the proof. $\square$

Corollary 4.41 does not apply when the B_0-minimal group is not stem. The group given in Example 4.35 is B_0-minimal and its center is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. The exponents of the upper central factors are, however, always bounded by p. This follows from the more general succeeding proposition. We use the notation $Z_i(G)$ for the i-th center of G, see [Hup67, Seite 259].

Proposition 4.42. *Let* G *be a* B_0*-minimal group. Then* $Z_2(G)$ *centralizes* $\Phi(G)$*.*

Proof. By Proposition 4.36, the group G is a p-group for some prime p. Suppose that $[Z_2(G), \Phi(G)] \neq 1$. Then there exist elements $x \in Z_2(G)$ and $y \in \Phi(G)$ with $[x, y] \neq 1$. By replacing y with its proper power, we may assume that the commutator $[x, y]$ is of order p. Invoking Lemma 4.38, we conclude that there exist elements $a, b \in G$ with $[x, y] = [a, b]$ and $x \curlywedge y \neq a \curlywedge b$. Hence $G = \langle a, b, x\rangle$ by B_0-minimality. As $y \in \Phi(G)$, we have $y = \prod_i w_i^p$ for some elements $w_i \in G$. Since $x \in Z_2(G)$, this implies $[x, y] = \prod_i [x, w_i]^p$ and $x \curlywedge y = \prod_i (x \curlywedge w_i)^p$. Moreover, we may consider the w_i's modulo $[G, G]$, since x commutes with $[G, G]$. Putting $w_i = x^{\gamma_i} a^{\alpha_i} b^{\beta_i}$ for some integers $\alpha_i, \beta_i, \gamma_i$, we have $[x, w_i] = [x, a^{\alpha_i} b^{\beta_i}]$ and similarly for the curly wedge. By collecting the factors, we obtain $[x, y] = [x, a^{p\alpha} b^{p\beta}]$ for some integers α, β. Suppose first that p divides α. Then $[x, a^{p\alpha}] = [x, a^{\alpha}]^p = [x^p, a^{\alpha}] = 1$ by Theorem 4.40. This implies $[x, y] = [x, b^{p\beta}]$. By an analogous argument, the prime p cannot divide β, since the commutator $[x, y]$ is not trivial. Let $\bar\beta$ be the multiplicative inverse of β modulo p and put $\tilde a = a^{\bar\beta}, \tilde b = b^{\beta}$. Then we have $[\tilde a, \tilde b] = [x, y] = [x, \tilde b^p] = [x^p, \tilde b]$ and similarly $\tilde a \curlywedge \tilde b = a \curlywedge b \neq x \curlywedge y = x^p \curlywedge \tilde b$. By B_0-minimality, this implies $G = \langle \tilde a, \tilde b\rangle$ with the commutator $[\tilde a, \tilde b]$ being central of order p in G. Hence the group G is of nilpotency class 2. We now have $[\tilde a^p, b] = [\tilde a, b]^p = 1$ and similarly $[\tilde b^p, a] = 1$, so the Frattini subgroup $\Phi(G)$ is contained in the center of G. This is a contradiction with $[x, y] \neq 1$. Hence the prime p cannot divide α, and the same argument shows that p cannot divide β. Let $\bar\alpha$ be the multiplicative inverse of α modulo p. Put $\tilde a = a^{\alpha}, \tilde b = b^{\bar\alpha}$. This gives $[x, y] = [x, \tilde a^p, \tilde b^{p\tilde\beta}]$ for some integer $\tilde\beta$, hence we may assume that $\alpha = 1$. Now put $\tilde a = ab$. We get $[x, y] = [x, \tilde a^p b^{p(\beta-1)}]$. By continuing in this manner, we degrade the exponent at the generator b to $\beta = 0$, reaching a final contradiction. ☐

Corollary 4.43. *Let G be a B_0-minimal group. Then $\exp Z_i(G)/Z_{i-1}(G) = p$ for all $i \geq 2$.*

Proof. It is a classical result [Hup67, Satz III.2.13] that the exponent of $Z_{i+1}(G)/Z_i(G)$ divides the exponent of $Z_i(G)/Z_{i-1}(G)$ for all i. Thus it suffices to prove that $\exp Z_2(G)/Z(G) = p$. To this end, let $x \in Z_2(G)$. For any $y \in G$, we have $[x^p, y] = [x, y]^p = [x, y^p] = 1$ by the preceding proposition. Hence $x^p \in Z(G)$ and the proof is complete. ☐

Corollary 4.41 can, however, be improved when the group is of small enough nilpotency class.

Corollary 4.44. *The center of a stem B_0-minimal group of nilpotency class 2 is of prime exponent.*

Proof. Let G be a stem B_0-minimal p-group of nilpotency class 2. We therefore have $Z(G) = [G, G]$. For any commutator $[x, y] \in G$, Proposition 4.42 gives $[x, y]^p = [x^p, y] = 1$, as required. ☐

4.4.3 B_0-minimal groups of nilpotency class 2

Using Corollary 4.44 together with Corollary 4.43, we classify all the B_0-minimal isoclinism families of nilpotency class 2. For later use, we also record the number of conjugacy classes of their stem group.

Theorem 4.45. *A B_0-minimal isoclinism family of nilpotency class 2 is determined by one of the following two stem p-groups:*

$$G_1 = \left\langle\ a,b,c,d \ \middle|\ \begin{array}{l} a^p = b^p = c^p = d^p = 1, \\ [a,b] = [c,d],\ [b,d] = [a,b]^{\varepsilon}[a,c]^{\omega},\ [a,d] = 1,\ \textit{class } 2 \end{array} \right\rangle,$$

$$G_2 = \left\langle\ a,b,c,d \ \middle|\ \begin{array}{l} a^p = b^p = c^p = d^p = 1, \\ [a,b] = [c,d],\ [a,c] = [a,d] = 1,\ \textit{class } 2 \end{array} \right\rangle,$$

where $\varepsilon = 1$ for $p = 2$ and $\varepsilon = 0$ for odd primes p, and ω is a generator of the group $(\mathbb{Z}/p\mathbb{Z})^{\times}$. The groups G_1 and G_2 are of order p^7, their Bogomolov multipliers are $\mathrm{B}_0(G_1) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, $\mathrm{B}_0(G_2) \cong \mathbb{Z}/p\mathbb{Z}$, and the number of their conjugacy classes equal $\mathrm{k}(G_1) = p(p^3 + 2p^2 - p - 1)$, $\mathrm{k}(G_2) = p^2(2p^2 + p - 2)$.

Proof of Theorem 4.45. Following Proposition 4.34 and Proposition 4.36, we may restrict ourselves to studying a stem B_0-minimal p-group G of nilpotency class 2. This immediately implies $Z(G) = [G,G]$, and it follows from Theorem 4.40 that the group G may be generated by 4 elements, say a,b,c,d, satisfying $[a,b] = [c,d]$. By Corollary 4.43 and Corollary 4.44, the exponents of both $[G,G]$ and $G/[G,G]$ equal to p. Furthermore, the derived subgroup of G is of rank at most $\binom{4}{2} - 1 = 5$, and $G/[G,G]$ is of rank at most 4. The order of the group G is therefore at most p^9.

Proposition 4.37 shows that no B_0-minimal isoclinism families of rank at most 6 are of nilpotency class 2. Hence G is of order at least p^7. Together with the above reasoning, this shows that G must be of Frattini rank precisely 4. Moreover, by possibly replacing G by a group isoclinic to it, we may assume without loss of generality that $a^p = b^p = c^p = d^p = 1$. The group G may therefore be regarded as a quotient of the group

$$K = \langle a,b,c,d \mid a^p = b^p = c^p = d^p = 1,\ [a,b] = [c,d],\ \text{class } 2\rangle\,,$$

which is of order p^9, nilpotency class 2, exponent p when p is odd, and has precisely one commutator relation. In the language of vector spaces from above, the cosets of elements $\{a,b,c,d\}$ form a basis of $G/[G,G]$, and G is determined by a subspace R of $\mathbb{F}_p^4 \wedge \mathbb{F}_p^4$ with the additional requirement $z_1 \wedge z_2 - z_3 \wedge z_4 \in R$.

Suppose first that G is of order precisely p^7. When $p = 2$, we invoke Proposition 4.37 to conclude that G belongs to either the family Φ_{30} or Φ_{31} due to the nilpotency class restriction. It is readily verified using the classification [JNO90] that the groups G_1 and G_2 given in the statement of the theorem are stem groups of these two families, respectively. Suppose now that p is odd. The p-groups of order p^7 have been classified by O'Brien and Vaughan-Lee [OVL05], the detailed notes on such groups of exponent p are available at [Vau01]. Following these, we see that the only stem groups of Frattini rank 4 and nilpotency class 2 are the groups whose corresponding Lie algebras are labeled as (7.16) to (7.20) in [Vau01]. In the groups arising from (7.16) and (7.17), the nontrivial commutators in the polycyclic presentations are all different elements of the polycyclic generating sequence. It follows from [Mor12 p5] that these groups have trivial Bogomolov multipliers. The remaining groups, arising from the algebras (7.18) to (7.20), are the following:

$$G_{18} = \left\langle\ a, b, c, d \ \middle|\ \begin{array}{l} a^p = b^p = c^p = d^p = 1, \\ [a, c] = [a, d] = 1, [a, b] = [c, d], \text{ class } 2 \end{array} \right\rangle,$$

$$G_{19} = \left\langle\ a, b, c, d \ \middle|\ \begin{array}{l} a^p = b^p = c^p = d^p = 1, \\ [a, d] = 1, [b, c] = [c, d], [b, d] = [a, c], \text{ class } 2 \end{array} \right\rangle,$$

$$G_{20} = \left\langle\ a, b, c, d \ \middle|\ \begin{array}{l} a^p = b^p = c^p = d^p = 1, \\ [a, d] = 1, [a, b] = [c, d], [b, d] = [a, c]^\omega, \text{ class } 2 \end{array} \right\rangle,$$

where ω is a generator of the multiplicative group of units of $\mathbb{Z}/p\mathbb{Z}$.

Let us first show that $\mathrm{B}_0(G_{19})$ is trivial. To this end, alter the presentation of G_{19} by replacing b with bd, which allows to assume $[b, c] = 1$. Translating the set of relations to $\mathbb{F}_p^4 \wedge \mathbb{F}_p^4$, the subspace of relations R consists of vectors of the form $\alpha_1(z_1 \wedge z_4) + \alpha_2(z_2 \wedge z_3) + \alpha_3(z_2 \wedge z_4 - z_1 \wedge z_3)$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}_p$. Plugging these into Plückers formula, we obtain the relation $-\alpha_3^2 = \alpha_1\alpha_2$. The solutions of this equation span the space $\mathbb{F}_p^3$, take for example the three independent solutions $(\alpha_1, \alpha_2, \alpha_3) \in \{(1, 0, 0), (0, 1, 0), (-1, 1, 1)\}$. It follows that $\langle \mathfrak{P} \cap R\rangle = R$, and so $\mathrm{B}_0(G) = 0$, as required.

We now turn to the group G_{18} and show that $\mathrm{B}_0(G_{18}) \cong \mathbb{Z}/p\mathbb{Z}$. As there are no groups of nilpotency class 2 and order at most p^6 with a nontrivial Bogomolov multiplier, this alone will immediately imply that G_{18} is a B_0-minimal group. Translating the set of relations to $\mathbb{F}_p^4 \wedge \mathbb{F}_p^4$, the subspace R consists of vectors of the form $\alpha_1(z_1 \wedge z_3) + \alpha_2(z_1 \wedge z_4) + \alpha_3(z_1 \wedge z_2 - z_3 \wedge z_4)$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}_p$. Plugging these into Plückers formula, we obtain the relation $\alpha_3^2 = 0$. The solutions of this equation form a subspace of the space $\mathbb{F}_p^3$ of dimension 2. It follows that $\mathrm{B}_0(G_{18}) = R/\langle \mathfrak{P} \cap R\rangle \cong \mathbb{Z}/p\mathbb{Z}$. It is also readily verified that $\mathrm{k}(G_{18}) = 2p^4 + p^3 - 2p^2$. In the statement of the theorem, the group G_{18} corresponds to G_2.

At last, we deal with the group G_{20}. Translating the set of relations to $\mathbb{F}_p^4 \wedge \mathbb{F}_p^4$, the subspace R consists of vectors of the form $\alpha_1(z_1 \wedge z_4) + \alpha_2(z_1 \wedge z_2 - z_3 \wedge z_4) + \alpha_3(z_2 \wedge z_4 - \omega z_1 \wedge z_3)$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}_p$. Plugging these into Plückers formula, we obtain the relation $\alpha_2^2 = \omega\alpha_3^2$. Since ω is a generator of the group of units of $\mathbb{F}_p$, it is not a square, and so the solutions of this equation form subspace of the space $\mathbb{F}_p^3$ of dimension 1. It follows that $\mathrm{B}_0(G_{18}) = R/\langle \mathfrak{P} \cap R\rangle \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. It is also readily verified that $\mathrm{k}(G_{20}) = p^4 + 2p^3 - p^2 - p$. In the statement of the theorem, the group G_{20} corresponds to G_1.

So far, we have dealt with the case when the B_0-minimal group G is of order at most p^7. Were G of order p^9, it would be isomorphic to the group K. By what we have shown so far, this group is not B_0-minimal, since it possesses proper quotients with nontrivial Bogomolov multipliers, namely both the groups G_1 and G_2. The only remaining option is for the group G to be of order p^8. Regarding G as a quotient of K, this amounts to precisely one additional commutator relation being imposed in K, i.e., one of the commutators in G may be expanded by the rest. By possibly permuting the generators, we may assume that this is the commutator $[b,d]$, so

$$[b,d]=[a,b]^{\alpha}[a,c]^{\beta}[a,d]^{\gamma}[b,c]^{\delta}$$

for some integers $\alpha,\beta,\gamma,\delta$. Replacing b by $ba^{-\gamma}$ and d by $dc^{-\delta}$, we may further assume $\gamma=\delta=0$.

For $p=2$, the above expansion reduces to only 4 possibilities. When $\alpha=\beta=0$, interchanging a with b and c with d shows that the group G possesses a proper quotient isomorphic to G_2. Next, when $\alpha=\beta=1$, the group G possesses a proper quotient isomorphic to G_1. In the case $\alpha=1$, $\beta=0$, replacing c by $b^{-1}c$ and a by ad enables us to rewrite the commutator relations to $[a,b]=[c,d]=1$. There are thus no commutator relations between the nontrivial commutators in G, so the Bogomolov multiplier of G is trivial. Finally, when $\alpha=0$, $\beta=1$, use [JNO90] to see that the group $G/\langle[a,d]\rangle$ belongs to the isoclinism family Φ_{31}, thus having a nontrivial Bogomolov multiplier by Proposition 4.37. This shows that G is not a B_0-minimal group in neither of these cases.

Now let p be odd. Translating the set of relations to $\mathbb{F}_p^4\wedge\mathbb{F}_p^4$, the subspace R consists of vectors of the form $\alpha_1(z_1\wedge z_2-z_3\wedge z_4)+\alpha_2(z_2\wedge z_4-\alpha z_1\wedge z_2-\beta z_1\wedge z_3)$ for some $\alpha_1,\alpha_2\in\mathbb{F}_p$. Note that R is contained in the subspace $\{Z_{14}=Z_{23}=0\}$. Now consider the space $\tilde{R}=R\oplus\mathbb{F}_p(z_1\wedge z_4)$. Observe that an element of $\tilde{R}$ belongs to $\mathfrak{P}$ if and only if its projection to R belongs to $\mathfrak{P}$. Hence $\langle\tilde{R}\cap\mathfrak{P}\rangle=\langle R\cap\mathfrak{P}\rangle\oplus\mathbb{F}_p(z_1\wedge z_4)$. Translating this equality back to the level of G, we have that $\mathrm{B}_0(G)\cong\mathrm{B}_0(G/\langle[a,d]\rangle)$. Therefore G can not be a B_0-minimal group. The proof is complete. $\square$

Note that Theorem 4.45 shows, in particular, that there exist B_0-minimal groups with noncyclic Bogomolov multipliers. We also record a corollary following from the proof of Theorem 4.45 here.

Corollary 4.46. *Let G be a p-group of order p^7 and nilpotency class 2. Then $\mathrm{B}_0(G)$ is nontrivial if and only if G belongs to one of the two isoclinism families given by Theorem 4.45. Moreover, the stem groups of these families are precisely the groups of minimal order that have nontrivial Bogomolov multipliers and are of nilpotency class 2.*

In general, there is no upper bound on the nilpotency class of a stem B_0-minimal group. We show this by means of constructing a stem B_0-minimal 2-group of order 2^n and nilpotency class $n-3$ for any $n\geq 6$. As we use Corollary 5.8 to do this, the example is provided in Section 5.2. On the other hand, the bound on the exponent of the center provided by Corollary 4.41 together with the bound on the number of generators given by Theorem 4.40 show that fixing the nilpotency class restricts the number of B_0-minimal isoclinism families.

Corollary 4.47. *Given a prime p and nonnegative integer c, there are only finitely many B_0-minimal isoclinism families containing a p-group of nilpotency class c.*

Proof. The exponent of a B_0-minimal p-group of class at most c is bounded above by p^{c+1} using Corollary 4.41 and Corollary 4.43. Since B_0-minimal groups may be generated by at most 4 elements by Theorem 4.40, each one is an epimorphic image of the free 4-generated c-nilpotent group of exponent p^{c+1}, which is a finite group. As a B_0-minimal isoclinism family is determined by its stem groups, the result follows. □

4.4.4 Breadth and width

Let us say something about the fact that the Frattini subgroup of a B_0-minimal group G is abelian.

The centralizer $C = C_G(\Phi(G))$ is of particular interest, as a classical result of Thompson, cf. [FT63], states that C is a critical group. The elements of G whose centralizer is a maximal subgroup of G are certainly contained in C. These elements have been studied by Mann in [Man06], where they are termed to have *minimal breadth*. We follow Mann in denoting by $\mathcal{M}(G)$ the subgroup of G generated by the elements of minimal breadth. Later on, we will be dealing separately with 2-groups. It is shown in [Man06, Theorem 5] that in this case, the nilpotency class of $\mathcal{M}(G)$ does not exceed 2, and that the group $\mathcal{M}(G)/Z(G)$ is abelian. We show that for B_0-minimal groups, the group $\mathcal{M}(G)$ is actually abelian.

Proposition 4.48. *Let G be a B_0-minimal 2-group. Then $\mathcal{M}(G)$ is abelian.*

Proof. Suppose that there exist elements $g, h \in G$ of minimal breadth with $[g,h] \neq 1$. Since the group $\mathcal{M}(G)/Z(G)$ is abelian, we have $[g,h] \in Z(G)$. Without loss of generality, we may assume that $[g,h]$ is of order 2, otherwise replace g by its power. Putting $z = [g,h]$ and applying Lemma 4.38, there exist elements $a, b \in G$ such that $G = \langle g,h,a,b \rangle$ and $a \curlywedge b \neq g \curlywedge h$ and $[a,b] = z$. Suppose that $[g,a] \neq 1$ and $[g,b] \neq 1$. Since g is of minimal breadth, we have $[g,a] = [g,b]$ and so the element $\tilde{a} = b^{-1}a$ centralizes g. This gives $z = [a,b] = [\tilde{a}, b]$ and similarly $\tilde{a} \curlywedge b = a \curlywedge b$. By B_0-minimality, it follows that $G = \langle g, h, \tilde{a}, b \rangle$ with $[g, \tilde{a}] = 1$. We may thus *a priori* assume that $[g,a] = 1$. Now suppose $[g,b] \neq 1$. Since g is of minimal breadth, we have $[g,b] = [g,h]$ and so the element hb^{-1} centralizes g. This gives $g \curlywedge h = g \curlywedge (hb^{-1})b = g \curlywedge b$. The product $(g \curlywedge b)(a \curlywedge b)^{-1}$ is thus a nontrivial element of $\mathrm{B}_0(G)$. By B_0-minimality, it follows that $G = \langle g,a,b \rangle$ with $[g,a] = 1$ and $[g,b] = [a,b] \in Z(G)$. Putting $\tilde{g} = ga^{-1}$, we have $[\tilde{g}, a] = 1$ and $[\tilde{g}, b] = 1$ with $G = \langle \tilde{g}, a, b \rangle$. This implies $[G,G] = \langle [a,b] \rangle \leq Z(G)$, so G is of nilpotency class 2. Therefore G belongs to one of the two families given in Theorem 4.45. The groups in both of these families have their derived subgroups isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. This is a contradiction, so we must have $[g,b] = 1$. Hence $G = \langle g,h,a,b \rangle$ with $[g,a] = [g,b] = 1$ and $[g,h] = z \in Z(G)$. By the same arguments applied to h instead of g, we also have $[h,a] = [h,b] = 1$. This implies $[G,G] = \langle [a,b] \rangle \leq Z(G)$, so G is again of class 2, giving a final contradiction. □

Another aspect of $\Phi(G)$ being abelian is that conjugation in $G \curlywedge G$ can at times be simplified, and that $G \curlywedge G$ is abelian.

Lemma 4.49. *Let G be a group with $\Phi(G)$ abelian. Let $g \curlywedge h \in G \curlywedge G$. Then $(g \curlywedge h)^f = g \curlywedge h$ for every $f \in \Phi(G)$. Moreover, if $[g,h] \in Z(G)$, then $(g \curlywedge h)^k = g \curlywedge h$ for every $k \in G$.*

Proof. Let E be a CP-cover of G, so that $[E,E] \cong G \curlywedge G$. Then $(g \curlywedge h)^f = [g^f, h^f]_E$. Since $[g^f, h^f]_G = [g,h]_G^f = [g,h]_G$, we have $[[g,h]_G, f]_G = 1$. As E is a CP-extension of G, it follows that $[[g,h]_E, f]_E = 1$, hence $[g^f, h^f]_E = [g,h]_E$ and the first claim holds. If we also have $[g,h]_G \in Z(G)$, then similary $[g^k, h^k]_G = [g,h]_G^k = [g,h]_G$, hence $[g^k, h^k]_E = [g,h]_E$. □

The *width* of an element $\omega \in G \wedge G$ is the smallest number n such that $\omega = \prod_{i=1}^n x_i \wedge y_i$ for some $x_i, y_i \in G$. We similarly define the width of elements of the curly exterior square $G \curlywedge G$.

Theorem 4.50. *Let G be a B_0-minimal group. Then there is an element $\omega_0 \in \mathrm{B}_0(G)$ of width 2 and every other nontrivial element $\omega \in \mathrm{B}_0(G)$ has width 2 modulo $\langle \omega_0 \rangle$.*

Proof. We may assume G is a stem group. Let E be a CP-cover of G and $\omega \in \mathrm{B}_0(G) \leq E$ a minimal commutator relation of G. It follows from the proof of Theorem 4.40 that G (and hence also E) may be generated by elements a, b, x, y such that $[a,b] = [x,y]$ and $x \in \gamma_{c-1}(G)$.

Suppose first that G is of nilpotency class 2. Then G is isoclinic to one of the two groups listed in Theorem 4.45. They both satisfy our claim.

Assume now that the nilpotency class of G is at least 3. Then $x \in [G,G] \leq \Phi(G)$, and so $G = \langle a, b, y \rangle$. By [Wil98, Proposition 4.3.2], the element $\omega \in [E,E] \cong G \curlywedge G$ may be written as $\omega = (a \curlywedge u_1)(b \curlywedge u_2)(y \curlywedge u_3)$ for some $u, v, w \in G$. Hence the width of ω is already bounded by 3. We now reduce this bound to 2 by rewriting ω.

Let $u_i = a^{\alpha_i} b^{\beta_i} y^{\gamma_i} f_i$ for some $0 \leq \alpha_i, \beta_i, \gamma_i \leq p-1$ and $f_i \in \Phi(G)$. Applying the lemma, we see that

$$\begin{aligned} a \curlywedge u_1 &= (a \curlywedge f_1)(a \curlywedge y^{\gamma_1})(a \curlywedge b^{\beta_1}), \\ b \curlywedge u_2 &= (b \curlywedge f_2)(b \curlywedge y^{\gamma_2})(b \curlywedge a^{\alpha_2}). \end{aligned}$$

We have $a \curlywedge y^{\gamma_1} = \prod_{i=0}^{\gamma_1 - 1} (a \curlywedge y)^{y^i} = \prod_{i=0}^{\gamma_1-1} (a[a, y^i] \curlywedge y) = (a \curlywedge y)^{\gamma_1} \cdot (y \curlywedge f_4)$ for some $f_4 \in \Phi(G)$, and similarly $b \curlywedge y^{\gamma_2} = (b \curlywedge y)^{\gamma_2} \cdot (y \curlywedge f_5)$ for some $f_5 \in \Phi(G)$. Therefore

$$(a \curlywedge y^{\gamma_1})(b \curlywedge y^{\gamma_2}) \cdot (y \curlywedge u_3) \;=\; (y \curlywedge a)^{-\gamma_1} (y \curlywedge b)^{-\gamma_2} \cdot (y \curlywedge u_3 f_4 f_5).$$

Furthermore, notice that for any $z \in G$ we have

$$\begin{aligned} (y \curlywedge az) &= (y \curlywedge z)(y \curlywedge a)^z \\ &= (y \curlywedge z)(y[y,z] \curlywedge a[a,z]) \\ &= (y \curlywedge z)(y \curlywedge a[a,z])([y,z] \curlywedge a[a,z]) \\ &= (y \curlywedge z[a,z])(y \curlywedge a)([y,z] \curlywedge a) \end{aligned}$$

Taking $z = a^{-1} u_3 f_4 f_5$, we obtain $(y \curlywedge a)^{-1} (y \curlywedge u_3 f_4 f_5) = (y \curlywedge u_4)(a \curlywedge f_6)$ for some $u_4 \in G$, $f_6 \in \Phi(G)$. Repeating the argument, we see that $(a \curlywedge y^{\gamma_1})(y \curlywedge u_3) = (y \curlywedge u_5)(a \curlywedge f_7)$

for some $u_5 \in G$, $f_7 \in \Phi(G)$. Applying the same reasoning with b instead of a gives $(a \curlywedge y^{\gamma_1})(b \curlywedge y^{\gamma_2}) \cdot (y \curlywedge u_3) = (y \curlywedge u_6)(a \curlywedge f_7)(b \curlywedge f_8)$ for some $u_6 \in G$, $f_7, f_8 \in \Phi(G)$. We therefore have

$$\begin{aligned} \omega &= (a \curlywedge u_1)(b \curlywedge u_2)(y \curlywedge u_3) \\ &= (a \curlywedge f_1)(b \curlywedge f_2) \cdot (a \curlywedge y^{\gamma_1})(b \curlywedge y^{\gamma_2}) \cdot (a \curlywedge b)^{\beta_1+\alpha_2} \cdot (y \curlywedge u_3) \\ &= (a \curlywedge f_1 f_7)(b \curlywedge f_2 f_8) \cdot (a \curlywedge b)^{\beta_1+\alpha_2} \cdot (y \curlywedge u_6). \end{aligned}$$

Now observe that for any $\phi_1, \phi_2 \in \Phi(G)$, we have

$$a\phi_2^{-1} \curlywedge b\phi_1 = (a \curlywedge b\phi_1)^{\phi_2^{-1}} (\phi_2^{-1} \curlywedge b\phi_1) = (a \curlywedge b)(a \curlywedge \phi_1)(b \curlywedge \phi_2).$$

Hence

$$\omega = (a(f_2 f_8)^{-1} \curlywedge b(f_1 f_7)) \cdot (a \curlywedge b)^{\beta_1+\alpha_2-1} \cdot (y \curlywedge u_6).$$

This shows that, modulo the relation $\sigma_1 = (a \curlywedge b)(x \curlywedge y)^{-1}$, ω is of width 2, since we have (using the fact that $x \in \Phi(G)$)

$$\begin{aligned} \omega &\equiv (a(f_2 f_8)^{-1} \curlywedge b(f_1 f_7)) \cdot (x \curlywedge y)^{\beta_1+\alpha_2-1} \cdot (y \curlywedge u_6) \\ &\equiv (a(f_2 f_8)^{-1} \curlywedge b(f_1 f_7)) \cdot (y \curlywedge u_6 x^{-\beta_1-\alpha_2+1}) \end{aligned}$$

modulo $\langle \sigma_1 \rangle$. □

We remark that if G is B_0-minimal, then picking any central subgroup $N = \langle z \rangle$ of order p in G shows that $\mathrm{B}_0(G) = \ker(J_N \to N)$ by Proposition 3.9. Note that J_N is elementary abelian. Whence $\mathrm{B}_0(G)$ can be generated by elements of the form $(a \curlywedge b)(c \curlywedge d)^{-1}$ for $[a, b] = [c, d] \in N$. So the minimal commutator relations can always be selected to be of the basic type.

5

Commuting probability bounds

The problem of triviality of B_0 is considered from the probabilistic point of view. As the structure of Bogomolov multipliers heavily depends on the structure of commuting pairs of elements of a given group, we inspect the probability that a randomly chosen pair of elements of a given group commute. The general principle is that the higher the probability of commuting, the more abelian-like the group is. We find the smallest bound on this probability that ensures the Bogomolov mutliplier is trivial. Applications are given. We also relate commuting probability to commutativity preserving extensions and show how the theory of CP covers can be used to produce structural bounds on the Bogomolov multiplier. These are used to bound the Bogomolov multiplier relative to the commuting probability.

This chapter is based on [JM15, JM].

5.1 Commuting probability

5.1.1 Commuting probability

Let G be a finite group. The quotient

$$\mathrm{cp}(G) = \frac{|\{(x, y) \in G \times G \mid [x, y] = 1\}|}{|G|^2}$$

is called the *commuting probability* of G. It is the probability that a randomly chosen pair of elements of G commute. The study of probabilistic group theory was pioneered by Erdös and Turán [ET68].

***Example* 5.1.** *We have* $\mathrm{cp}(Q_8) = 5/8$ *and* $\mathrm{cp}(A_4) = 1/3$.

The behavior of the function cp is quite irregular. Some conjectures concerning the set of all possible commuting probabilities $\mathrm{im}\,\mathrm{cp}$ were made by Joseph [Jos77]. Some of these have been subsequently proved; for example, it is shown in [Ebe15, Heg13] that the limit points of the image of cp are all rational, and that if ℓ is a limit point of $\mathrm{im}\,\mathrm{cp}$, then there is an $\varepsilon > 0$ such that $\mathrm{im}\,\mathrm{cp} \cap (\ell - \varepsilon, \ell) = \emptyset$.

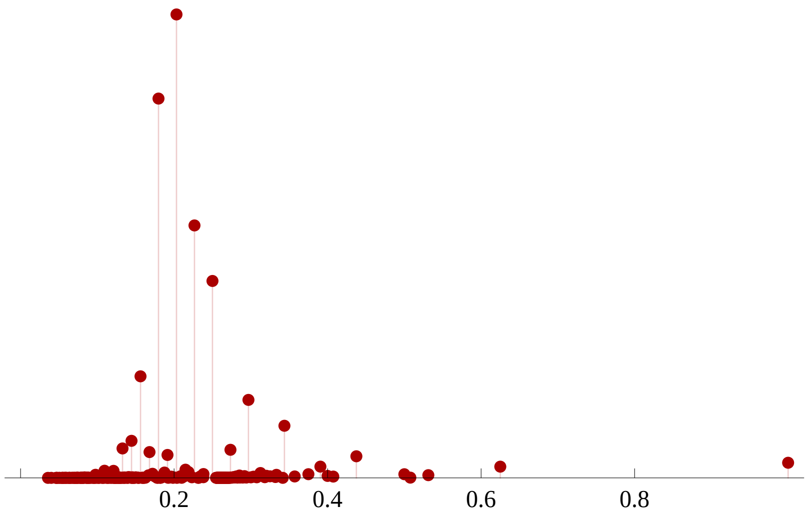

Figure 5.1: Number of groups of order at most 256 with a specific commuting probability.

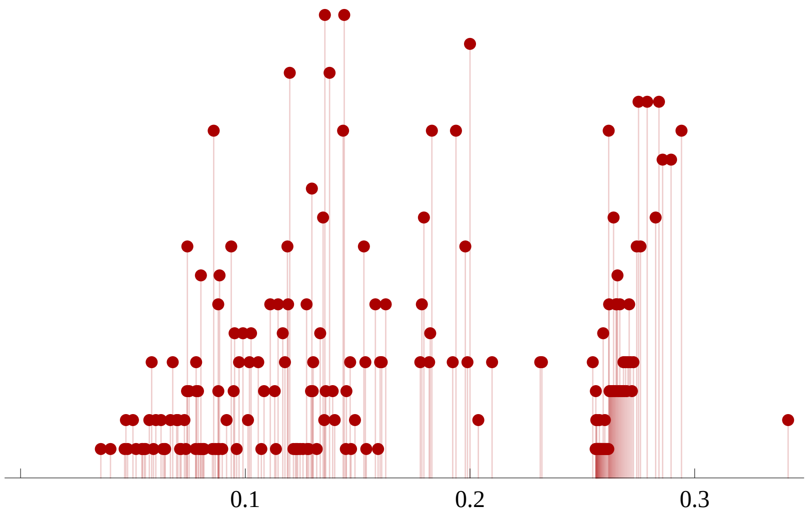

Figure 5.2: Number of groups of order at most 256 with a specific commuting probability, zoomed.

5.1.2 Basic properties

There is a simple formula for commuting probability $\mathrm{cp}(G)$ in terms of the number of conjugacy classes $\mathrm{k}(G)$ of a group G.

Theorem 5.2 ([ET68])**.** *Let G be a finite group. Then $\mathrm{cp}(G) = \mathrm{k}(G)/|G|$.*

Proof. Let $\mathcal{C}$ be a set of representatives of conjugacy classes of G. We have

$$\mathrm{cp}(G) = \frac{\sum_{x \in G} |C_G(x)|}{|G|^2} = \frac{\sum_{x \in \mathcal{C}} |x^G||C_G(x)|}{|G|^2} = \frac{\mathrm{k}(G)|G|}{|G|^2},$$

whence the result. □

Corollary 5.3. *Let G and H be finite groups. Then $\mathrm{cp}(G \times H) = \mathrm{cp}(G)\,\mathrm{cp}(H)$.*

Corollary 5.4. *Let G and H be isoclinic finite groups. Then $\mathrm{cp}(G) = \mathrm{cp}(H)$.*

We also emphasize that taking subgroups of quotients increases commuting probability.

Proposition 5.5. *Let G be a finite group, H a subgroup of G and N a normal subgroup of G. Then $\mathrm{cp}(G) \leq \mathrm{cp}(H)$, and $\mathrm{cp}(G) \leq \mathrm{cp}(G/N)\,\mathrm{cp}(N)$.*

Proof. Take any $x \in G$. Since $|C_G(x) : C_H(x)| \leq |G : H|$, it follows that $|C_G(x)| \leq |G : H||C_H(x)|$. Therefore

$$\mathrm{cp}(G) = \frac{\sum_{x \in G} |C_G(x)|}{|G|^2} \leq \frac{|G : H| \sum_{x \in G} |C_H(x)|}{|G|^2}.$$

Now, since an element $y \in G$ belongs to $C_H(x)$ if and only if x belongs to $C_G(y)$, we have

$$\mathrm{cp}(G) = \frac{|G : H| \sum_{y \in H} |C_G(y)|}{|G|^2} \leq \frac{|G : H|^2 \sum_{y \in H} |C_H(y)|}{|G|^2},$$

and the first claim follows. The second claim is a consequence of a well-known result of Gallagher [Gal70]. □

5.1.3 Bounding commuting probability

The number $\mathrm{cp}(G)$ may be thought of as a type of measure of how close the group G is to being abelian. In this sense, bounding $\mathrm{cp}(G)$ away from zero ensures abelian-like properties of G. The first result illustrating this general principle was the following.

Theorem 5.6 ([Gus73])**.** *Let G be a finite group. If $\mathrm{cp}(G) > 5/8$, then G is abelian.*

Proof. Consider the class equation for G. Bounding the size of every nontrivial orbit from above by 2, we obtain $|G| \geq |Z(G)| + 2(\mathrm{k}(G) - |Z(G)|)$. Solving for $\mathrm{k}(G)$ yields $\mathrm{k}(G) \leq (|G| + |Z(G)|)/2$. If G is not abelian, then $|G/Z(G)| \geq 4$, and hence $\mathrm{k}(G) \leq 5|G|/8$. □

There are also subtler results regarding the structure of a group whose commuting probability is bounded from below. Lescot showed that if $\mathrm{cp}(G) > 1/2$, then G is nilpotent [Les95]. We also have, for example, that $\mathrm{cp}(G)$ is always smaller than $|G : \mathrm{Fit}(G)|^{-1/2}$ by [GR06].

5.2 Absolute bounds

5.2.1 The $1/4$ bound

Let G be a finite group. We give an explicit lower bound for commuting probability of G that ensures triviality of its nonuniversal commutator relations. This is first done for p-groups.

Theorem 5.7. *Let G be a finite p-group. If $\mathrm{cp}(G) > (2p^2 + p - 2)/p^5$, then $\mathrm{B}_0(G)$ is trivial.*

It is then easy to obtain a global bound applicable to all finite groups.

Corollary 5.8. *Let G be a finite group. If $\mathrm{cp}(G) > 1/4$, then $\mathrm{B}_0(G)$ is trivial.*

Proof. Let p be a prime dividing the order of G. The p-part of $\mathrm{B}_0(G)$ embeds into $\mathrm{B}_0(S)$, where S is a Sylow p-subgroup of G. At the same time, we have $\mathrm{cp}(S) \geq \mathrm{cp}(G) > 1/4$, which gives $\mathrm{B}_0(S) = 0$ by Theorem 5.7. Hence $\mathrm{B}_0(G) = 0$. □

The bound given by both Theorem 5.7 and Corollary 5.8 is sharp, as shown by the existence of groups given in Theorem 4.45 with commuting probability equal to $(2p^2 + p - 2)/p^5$ and a nontrivial Bogomolov multiplier. We also note that no sensible converse of neither Theorem 5.7 nor Corollary 5.8 holds. As an example, let G be a noncommutative group with $\mathrm{B}_0(G) = 0$, and take G_n to be the direct product of n copies of G. It is clear that $\mathrm{cp}(G_n) = \mathrm{cp}(G)^n$, which tends to 0 with large n, and $\mathrm{B}_0(G) = 0$. So there exist groups with arbitrarily small commuting probabilities yet trivial Bogomolov multipliers.

Our proof is based on an intricate step-by-step argument that we split into subsections.

5.2.2 Proof: Reduction and small cases

Assume that G is a p-group of the smallest possible order satisfying $\mathrm{cp}(G) > (2p^2 + p - 2)/p^5$ and $\mathrm{B}_0(G) \neq 0$. As both commuting probability and the Bogomolov multiplier are isoclinism invariants, we can assume without loss of generality that G is a stem group. The commuting probability of a subgroup or a quotient of G exceeds $(2p^2 + p - 2)/p^5$, so all proper subgroups and quotients of G have a trivial multiplier by minimality of G. This implies that G is a stem B_0-minimal group.

We first deal with the cases when G is of small order in the following lemma.

Lemma 5.9. *Let G be a finite p-group belonging to an isoclinism family of rank at most* 6 *for odd p, or at most* 7 *for $p = 2$. If* $\mathrm{cp}(G) > (2p^2 + p - 2)/p^5$*, then* $\mathrm{B}_0(G)$ *is trivial.*

Proof. It suffices to verify the lemma for the isoclinism families of groups with nontrivial multipliers given in Subsection 4.3.2 and [CM13]. For odd primes, commuting probabilities of such families are given in [Jam80, Table 4.1]. The bound $(2p^2 + p - 2)/p^5$ is attained with the families Φ_{10}, Φ_{18} and Φ_{20}, while the rest of them have smaller commuting probabilities. Similarly, commuting probabilities of such families of 2-groups of rank at most 7 are given in [JNO90, Table II]. The bound 1/4 is attained with the families Φ_{16} and Φ_{31}, while the rest indeed all have smaller commuting probabilities. This proves the lemma. □

Suppose that G is of nilpotency class 2. By Theorem 4.45, G belongs to one of the isoclinism families given by the two stem groups in the theorem. The groups in both of these families have commuting probability at most $(2p^2 + p - 2)/p^5$, which is in conflict with the restriction on $\mathrm{cp}(G)$.

So we may assume from now on that the group G is of nilpotency class at least 3. Hence G has an abelian Frattini subgroup of index at most p^3 by Theorem 4.40. Note also that $|G : \Phi(G)| \geq p^2$, as G is not cyclic. We split the rest of the proof according to whether the minimal number of generators of G equals three or two.

5.2.3 Proof: Rank 3

Suppose that $|G : \Phi(G)| = p^3$. In light of Lemma 4.38, the generators g_1, g_2, g_3 of G may be chosen in such a way that the commutator $[g_1, f] = [g_3, g_2]$ is central and of order p for some $f \in \gamma_{c-1}(G) \leq \Phi(G)$. Put

$$z = \min\{|G : C_G(g_1^k \phi)| \mid 0 < k < p,\, \phi \in \langle g_2, g_3, \Phi(G)\rangle\}$$

and let (k_1, ϕ_1) be the pair at which the minimum is attained. We have $\phi_1 \equiv g_2^{\alpha_2} g_3^{\alpha_3}$ modulo $\Phi(G)$ for some $0 \leq \alpha_2, \alpha_3 < p$. After possibly replacing g_2 by $g_2^{\alpha_2} g_3^{\alpha_3}$, we may assume that not both α_2, α_3 are nonzero, hence $\alpha_2 = 0$ and $\alpha_3 = 1$ without loss of generality. Replacing g_1 by $\tilde{g}_1 = g_1^{k_1} g_3$, f by $\tilde{f} = f^{k_1}$, and g_2 by $\tilde{g}_2 = g_2^{k_1} \tilde{f}$, we still have $G = \langle \tilde{g}_1, \tilde{g}_2, g_3\rangle$ and $[\tilde{g}_1, \tilde{f}] = [g_1, f]^{k_1} [g_3, \tilde{f}] = [g_3, g_2^{k_1}][g_3, \tilde{f}] = [g_3, \tilde{g}_2]$ since $f \in \gamma_{c-1}(G)$. The minimum $\min\{|G : C_G(\tilde{g}_1 \phi)| \mid \phi \in \langle \tilde{g}_2, g_3, \Phi(G)\rangle\}$ is, however, now attained at $(1, g_3^{-1}\phi_1)$ with $g_3^{-1}\phi_1 \in \Phi(G)$. We may therefore assume that $k_1 = 1$ and $\phi_1 \in \Phi(G)$. Moreover, replacing g_1 by $\tilde{g}_1 = g_1 \phi_1$, we have both $[\tilde{g}_1, f] = [g_1, f] = [g_3, g_2]$ and $|G : C_G(\tilde{g}_1)| = z$, so we may actually assume that $\phi_1 = 1$. Next, put $x = \min\{|G : C_G(\phi)| \mid \phi \in \langle g_2, g_3, \Phi(G)\rangle \setminus \Phi(G)\}$ with the minimum being attained at the pair $g_2^\alpha g_3^\beta \phi_0$ with $\phi_0 \in \Phi(G)$. Replace the generators g_2, g_3 by setting $\tilde{g}_3 = g_2^\alpha g_3^\beta$ and choosing an element $\tilde{g}_2$ arbitrarily as long as $\langle g_2, g_3, \Phi(G)\rangle = \langle \tilde{g}_2, \tilde{g}_3, \Phi(G)\rangle$ and $[g_3, g_2] = [g_1, f^\kappa]$ for some κ. This enables us to assume that the minimum $\min\{|G : C_G(\phi)| \mid \phi \in \langle g_2, g_3, \Phi(G)\rangle \setminus \Phi(G)\}$ is attained at $g_3 \phi_0$ for some $\phi_0 \in \Phi(G)$. Lastly, put

$$y = \min\{|G : C_G(g_2^k \phi)| \mid 0 < k < p,\, \phi \in \langle g_3, \Phi(G)\rangle\}$$

with the minimum being attained at the pair (k_2, ϕ_2). Writing $\phi_2 \equiv g_3^{\alpha_3}$ modulo $\Phi(G)$ and then replacing g_2 by $\tilde{g}_2 = g_2^{k_2} g_3^{\alpha_3}$ and f by $\tilde{f} = f^{k_2}$ yields $G = \langle g_1, \tilde{g}_2, g_3 \rangle$ and $[g_1, \tilde{f}] = [g_3, g_2]^{k_2} = [g_3, \tilde{g}_2]$. We may thus *a priori* assume that $k_2 = 1$ and $\phi_2 \in \Phi(G)$. Note also that

$$x = \min\{|G : C_G(g_3^k \phi)| \mid 0 < k < p,\ \phi \in \Phi(G)\}$$

and the minimum is attained at $(1, \phi_0)$. Moreover, by the very construction of g_3, we have $x \leq y$.

When any of the numbers x, y, z equals p, the centralizer of the corresponding element is a maximal subgroup of G, and thus contains $\Phi(G)$. In the case $z = p$, this implies $[g_1, f] = 1$, which is impossible, so we must have $z \geq p^2$. As a consequence, $\mathcal{M}(G) \leq \langle g_2, g_3, \Phi(G) \rangle$. When $p = 2$, the group $\mathcal{M}(G)$ is abelian by Proposition 4.48, and so the factor group $G/\mathcal{M}(G)$ is not cyclic. This implies that not both g_2 and g_3 belong to $\mathcal{M}(G)$ in this case.

Let us first show that the case $z = p^2$ is only possible for groups of small orders, which have been dealt with at the beginning of the proof.

Lemma. *If $z = p^2$, then $|G| \leq p^6$ for odd p, and $|G| \leq 2^7$ for $p = 2$.*

Proof. We first show that the assumption $z = p^2$ implies that G is of nilpotency class 3. Observe that $|[g_1, G]| = p^2$. As the group G is of nilpotency class at least 3, not both the commutators $[g_2, g_1]$ and $[g_3, g_1]$ belong to $\gamma_3(G)$. By possibly replacing g_2 by g_3 (note that by doing so, we lose the assumption $x \leq y$, but we will not be needing it in this step), we may assume that $[g_3, g_1] \notin \gamma_3(G)$. Hence $[g_1, G] = \{[g_1, g_3^{\alpha} f^{\beta}] \mid 0 \leq \alpha, \beta < p\}$. It now follows that for any $g \in \gamma_2(G)$, we have $[g_1, g] \in \{[g_1, f^{\beta}] \mid 0 \leq \beta < p\}$, because the commutator $[g_1, g]$ itself belongs to $\gamma_3(G)$. This implies $[g_1, \gamma_2(G), G] = 1$. If the nilpotency class of G it at least 4, we have $[\gamma_{c-1}(G), g_1] = [\gamma_{c-2}(G), G, g_1] = [\gamma_{c-2}(G), g_1, G]$ as the group G is metabelian by Theorem 4.40. This gives $[\gamma_{c-1}(G), g_1] \leq [g_1, \gamma_2(G), G] = 1$, a contradiction with $[g_1, f] \neq 1$. Hence G must be of nilpotency class 3.

Consider the case when $p = 2$ first. As the Frattini subgroup of G is abelian, we have $[g_1^2, g_3^2] = 1$, which in turn gives $[g_1^4, g_3] = [g_1, g_3]^4 [g_1, g_3, g_1]^2 = 1$, and similarly $[g_1^4, g_2] = [g_3^4, g_1] = 1$. Hence g_1^4, g_2^4, g_3^4 are all central in G, and therefore belong to $[G, G]$ as the group G is stem. The factor group $\gamma_2(G)/\gamma_3(G)$ is generated by the commutator $[g_3, g_1]$, and we either have $[g_2, g_1] = [g_3, g_1]$ or $[g_2, g_1] \in \gamma_3(G)$, since $|[g_1, G]| = z = 4$ and thus $[g_1, G] = \{1, [g_1, f], [g_1, g_3], [g_1, g_3 f]\}$. Moreover, we can assume that $f = [g_3, g_1]$. Note that $[g_3^2, g_1] \in \gamma_3(G)$, implying $[g_3, g_1]^2 \in \gamma_3(G)$ and therefore $|\gamma_2(G)/\gamma_3(G)| = 2$. The group $\gamma_3(G)$ is generated by the commutators $[g_3, g_1, g_1]$, $[g_3, g_1, g_2]$ and $[g_3, g_1, g_3]$, all being of order at most 2. If $[g_2, g_1] \in \gamma_3(G)$, we have $[g_3, g_1, g_2] = [g_3, g_1]^{-1} [g_3^{g_2}, g_1^{g_2}] = 1$, and if $[g_2, g_1] \notin \gamma_3(G)$, then we have $[g_3, g_1]^{g_2} = [g_3, g_1 [g_1, g_2]] = [g_3, g_1 [g_3, g_1]] = [g_3, g_1][g_3, g_1, g_3]$, which gives $[g_3, g_1, g_2] = [g_3, g_1, g_3]$. Replacing g_2 by $\tilde{g}_2 = g_2 g_3$ therefore enables us to assume $[g_3, g_1, g_2] = 1$. This shows that $\gamma_3(G)$ is of order at most 4, and the Hall-Witt identity [Hup67, Satz III.1.4] gives $[g_2, g_1, g_3] = 1$. Note that $[g_3^2, g_1] \in \gamma_3(G)$ gives $[g_3^2, g_1] = [g_3, g_1, g_1]^k$ for some $k \in \{0, 1\}$, hence $g_3^2[g_3, g_1]^{-k}$ is central in G. Therefore $g_3^2 \in [G, G]$ as G is a stem group. The same reasoning shows that $[g_2^2[g_3, g_1]^k, g_1] = 1$ for some $k \in \{0, 1\}$. If $k = 0$, then g_2^2 is central in G and hence belongs to $[G, G]$. This gives $|G| \leq 2^7$, a contradiction. Hence $k = 1$ and we have $[g_2^2, g_1] = [g_3, g_1, g_1]$. This further implies $[g_2^2, g_1] = [g_2, g_1]^2[g_2, g_1, g_2] = [g_3, g_1]^2[g_3, g_1, g_2] = [g_3, g_1]^2$, hence $[g_3, g_1^2] = [g_3, g_1]^2[g_3, g_1, g_1] = [g_3, g_1]^2[g_2^2, g_1] = 1$. It follows from here that $[g_2, g_1^2] = [g_2, g_1]^2[g_2, g_1, g_1] = [g_3, g_1]^2[g_3, g_1, g_1] = [g_3, g_1^2] = 1$, and so g_1^2 is central in G. This finally gives $|G| \leq 2^7$.

Suppose now that p is odd. Commutators and powers relate to give the equality $[g_3^p, g_1] = [g_3, g_1]^p[g_3, g_1, g_3]^{\binom{p}{2}} = [g_3, g_1]^p = [g_3, g_1^p]$. Assuming $g_3^p \curlywedge g_1 \neq g_3 \curlywedge g_1^p$ and invoking B_0-minimality implies $G = \langle g_1, g_3 \rangle$, a contradiction. Hence $g_3^p \curlywedge g_1 = g_3 \curlywedge g_1^p$. Note that $[g_3, g_1]^p$ belongs to $\gamma_3(G)$, so we must have $[g_3, g_1]^p = [g_1, f^k]$ for some k. Assuming $g_3^p \curlywedge g_1 \neq g_1 \curlywedge f^k$ and invoking B_0-minimality gives $G = \langle g_1, g_3^p, f^k \rangle = \langle g_1 \rangle$, which is impossible. Hence we also have $g_3^p \curlywedge g_1 = g_1 \curlywedge f^k$. Recall, however, that $g_1 \curlywedge f \neq g_3 \curlywedge g_2$, which gives $g_3 \curlywedge g_1^p \neq g_3^k \curlywedge g_2$ whenever $k > 0$. Referring to B_0-minimality, a contradiction is obtained, showing that $k = 0$ and hence $[g_3, g_1]^p = 1$. An analogous argument shows that $[g_2, g_1]^p = 1$. The elements g_1^p, g_2^p, g_3^p are therefore all central in G, which implies that $|G/[G, G]| = p^3$ as G is a stem group. Now consider the commutator $[g_2, g_1]$. Should it belong to $\gamma_3(G)$, we have $\gamma_2(G) = \langle [g_3, g_1], [g_3, g_1, g_1], [g_3, g_1, g_3] \rangle$, since $[g_3, g_1, g_2] = 1$ by the Hall-Witt identity. The latter gives the bound $|G| = |G/[G, G]| \cdot |[G, G]| \leq p^6$, a contradiction. Now assume that $[g_2, g_1]$ does not belong to $\gamma_3(G)$. By the restriction $|[g_1, G]| = p^2$, we must have $[g_2, g_1] \equiv [g_3^k, g_1]$ modulo $\gamma_3(G)$ for some $k > 0$. Hence $|\gamma_2(G)/\gamma_3(G)| = p$ and $\gamma_3(G) = \langle [g_3, g_1, g_1], [g_3, g_1, g_2], [g_3, g_1, g_3] \rangle$. As in the case when $p = 2$, we now have $[g_3, g_1]^{g_2} = [g_3, g_1[g_1, g_2]] = [g_3, g_1[g_1, g_3^k]] = [g_3, g_1, g_3]^{-k}[g_3, g_1]$, which furthermore gives $[g_3, g_1, g_2] = [g_3, g_1, g_3]^{-k}$. All-in-all, we obtain the bound $|\gamma_3(G)| \leq p^2$ and therefore $|G| \leq p^6$. □

Assume now that $z \geq p^3$. Applying the restriction on commuting probability of G reduces our claim to just one special case.

Lemma. *We have $x = p$, $y = p^2$, $z = p^3$.*

Proof. We count the number of conjugacy classes in G with respect to the generating set g_1, g_2, g_3. The central elements $Z(G)$ are of class size 1, and the remaining elements of $\Phi(G)$ are of class size at least p. Any other element of G may be written as a product of powers of g_1, g_2, g_3 and an element belonging to $\Phi(G)$. These are of class size at least x, y, z, depending on the first nontrivial appearance of one of the generators. Summing up, we have

$$\mathrm{k}(G) \leq |Z(G)| + (|\Phi(G)| - |Z(G)|)/p + ((p-1)/x + p(p-1)/y + p^2(p-1)/z)|\Phi(G)|.$$

Note that since G is a 3-generated stem group of nilpotency class at least 3, we have $|G/Z(G)| = |G/[G, G]| \cdot |[G, G]/Z(G)| \geq p^4$. Applying this inequality, the commuting probability bound $(2p^2 + p - 2)/p^5 < \mathrm{cp}(G) = \mathrm{k}(G)/|G|$, and the information on the number of generators $|G : \Phi(G)| = p^3$, we obtain

$$(2p + 1)/p^4 < 1/p^2x + 1/py + 1/z. \tag{5.1}$$

Assume first that $x \geq p^2$. We thus also have $y \geq p^2$, and inequality (5.1) gives $z < p^3$, which is impossible. So we must have $x = p$. In particular, the generator g_3 centralizes $\Phi(G)$. We may thus replace g_2 by $\tilde{g}_2 = g_2\phi_2$ and henceforth assume that $|[g_2, G]| = y$. When $p = 2$, not both g_2 and g_3 belong to $\mathcal{M}(G)$, so we have $y \geq 4$ in this case. For odd primes p, assuming $y = p$ makes it possible to replace g_3 by $g_3\phi_3$ and hence assume $|G : C_G(g_3)| = p$. This implies that the commutators $[g_1, g_2]$, $[g_3, g_1]$ and $[g_3, g_2]$ all belong to $\gamma_c(G)$, which restricts the nilpotency class of G to at most 2, a contradiction. We therefore have $y = |[g_2, G]| \geq p^2$. Inequality (5.1) now gives $z < p^4$, which is only possible for $z = p^3$. Plugging this value in (5.1), we obtain $y < p^3$, so we must also have $y = p^2$. □

We are thus left with the case $x = p$, $y = |[g_2, G]| = p^2$, and $z = |[g_1, G]| = p^3$. These restrictions give a good bound on the nilpotency class of G.

Lemma. *The nilpotency class of G is at most* 4.

Proof. The commutator $[g_3, g_2]$ is central in G, and we have $[g_3, g_1, g_2] = 1$ by the Hall-Witt identity. This implies that $[g_3, g_1, g_1]^{g_2} = [g_3, g_1, g_1[g_1, g_2]] = [g_3, g_1, g_1]$, hence $[g_3, g_1, g_1, g_2] = 1$. The same reasoning gives $[g_3, g_1, g_1, g_1, g_2] = 1$. Note that we must have $[g_3, g_1, g_1, g_1, g_1] = 1$ since $|[g_1, G]| = p^3$. The commutator $[g_3, g_1, g_1, g_1]$ is therefore central in G, and the same argument applies to $[g_2, g_1, g_1, g_1]$. Note also that since $|[g_2, G]| = p^2$, the commutator $[g_2, g_1, g_2]$ is equal to a power of $[g_3, g_2]$, hence central in G. All together, this shows that all basic commutators of length 4 are central in G, which implies that G is of nilpotency class at most 4. □

We will also require the following result.

Lemma. *The element g_1^p is central in G.*

Proof. The restriction $|[g_2, G]| = p^2$ implies that $[g_2, g_1^p] = [g_3, g_2]^k$ for some k. Assuming $g_2 \curlywedge g_1^p \neq g_3^k \curlywedge g_2$ and invoking B_0-minimality gives $G = \langle g_2, g_3 \rangle$, which is impossible. Hence $g_2 \curlywedge g_1^p = g_3^k \curlywedge g_2$. When $k > 0$, this gives $g_2 \curlywedge g_1^p \neq g_1^k \curlywedge f$, hence $G = \langle g_2, g_1 \rangle$, a contradiction. Therefore $k = 0$ and we conclude $[g_2, g_1^p] = 1$, so g_1^p is central in G. □

The final step of the proof is based on whether or not the commutator $[g_2, g_1]$ belongs to $\gamma_3(G)$. In both cases, we reduce the claim to groups of smallish orders that have been considered above.

Lemma. *If $[g_2, g_1] \in \gamma_3(G)$, then $|G| \leq p^6$ for odd p, and $|G| \leq 2^7$ for $p = 2$.*

Proof. Suppose $[g_2, g_1] \in \gamma_3(G)$. When p is odd, this restriction is used to obtain $[g_2^p, g_1] = [g_2, g_1]^p [g_2, g_1, g_2]^{\binom{p}{2}} = [g_2, g_1]^p = [g_2, g_1^p] = 1$, showing that the element g_2^p is central in G. Furthermore, we have $\gamma_2(G)/\gamma_3(G) = \langle [g_3, g_1] \rangle$, $\gamma_3(G)/\gamma_4(G) =$

$\langle [g_3, g_1, g_1] \rangle$, and $\gamma_4(G) = \langle [g_3, g_1, g_1, g_1] \rangle$, with all of the factor group being of order p. When the nilpotency class of G equals 3, we thus obtain the bound $|G| = |G/[G,G]| \cdot |\gamma_2(G)/\gamma_3(G)| \cdot |\gamma_3(G)| \leq p^6$ for odd p and $|G| \leq 2^7$ for $p = 2$. Now let $[g_3, g_1, g_1, g_1] \neq 1$ and consider the commutator $[g_3^p, g_1]$. Since $|[g_1, G]| = p^3$, we have $[g_1, g_3^p] = [[g_3, g_1]^k [g_3, g_1, g_1]^l, g_1]$ for some k, l. This shows that $g_3^p [g_3, g_1]^{-k} [g_3, g_1, g_1]^{-l}$ is central in G. Since G is a stem group, we conclude that $|G/[G,G]| = p^3$ when p is odd, and $|G/[G,G]| \leq 2^4$ when $p = 2$. Applying the same bound as above gives $|G| \leq p^6$ for odd p and $|G| \leq 2^7$ for $p = 2$. □

Lemma. *If $[g_2, g_1] \notin \gamma_3(G)$, then $|G| \leq p^6$.*

Proof. Assume that $[g_2, g_1] \notin \gamma_3(G)$. Consider the commutator $[g_2, g]$ for some $g \in \gamma_2(G)$. Since $|[g_2, G]| = p^2$ and $[g_2, g] \in \gamma_3(G)$, we have $[g_2, g] = [g_3, g_2]^k$ for some k. Assuming $g_2 \curlywedge g \neq g_3^k \curlywedge g_2$ and invoking B_0-minimality gives $G = \langle g_2, g_3 \rangle$, a contradiction. Hence $g_2 \curlywedge g = g_3^k \curlywedge g_2$, implying $g_2 \curlywedge g \neq g_1^k \curlywedge f$ whenever $k > 0$, and it follows from here by B_0-minimality that $G = \langle g_2, g_1 \rangle$, another contradiction. We therefore have $[g_2, g] = 1$, that is $[g_2, \gamma_2(G)] = 1$. Now consider the commutator $[g_2^p, g_1]$. Since $[g_2^p, g_1] \equiv [g_2, g_1]^p \equiv [g_2, g_1^p] \equiv 1$ modulo $\gamma_3(G)$, we have $[g_2^p, g_1] = [g, g_1]$ for some $g \in [G,G]$. As G is a stem group, this implies that $g_2^p \in [G,G]$. The same reasoning applied to g_3 shows that $g_3^p \in [G,G]$. Hence $|G/[G,G]| = p^3$. At the same time, the derived subgroup $[G,G]$ is generated by the commutators $[g_3, g_1], [g_2, g_1], [g_3, g_1, g_1], [g_2, g_1, g_1], [g_3, g_1, g_1, g_1], [g_2, g_1, g_1, g_1]$. By $[g_2, \gamma_2(G)] = 1$, we have $[G,G] = [g_1, G]$ and therefore $|[G,G]| = p^3$. All together, the bound $|G| \leq p^6$ is obtained. □

5.2.4 Proof: Rank 2

Suppose that $|G : \Phi(G)| = p^2$. Let g_1 and g_2 be the two generators of G, satisfying $[g_1, f] = [g_3, g_2]$ for some $f \in \gamma_{c-1}(G)$. As before, put $y = \min\{|G : C_G(\phi)| \mid \phi \in \langle g_1, g_2, \Phi(G) \rangle \backslash \Phi(G)\}$. After possibly replacing the generators, we may assume

$$y = \min\{|G : C_G(g_2^k \phi)| \mid 0 < k < p,\, \phi \in \Phi(G)\} = |G : C_G(g_2)|.$$

Additionally put

$$z = \min\{|G : C_G(g_1^k \phi)| \mid 0 < k < p,\, \phi \in \langle g_2, \Phi(G) \rangle\}$$

with the minimum being attained at the pair $(1, 1)$ after possibly replacing g_1 and g_3 just as in the case when $|G : \Phi(G)| = p^3$. Note that we have $y \leq z$ by construction. When $y = p$, the subgroup $\langle g_2, \Phi(G) \rangle$ is a maximal abelian subgroup of G, which implies $\mathrm{B}_0(G) = 0$, a contradiction. Hence $z, y \geq p^2$.

Applying the restriction on commuting probability of G again reduces our claim to a special case.

Lemma. *We have $y = p^2$ and $z \leq p^3$.*

Proof. We count the number of conjugacy classes in G. In doing so, we may assume $|G/Z(G)| \geq p^4$. To see this, suppose for the sake of contradiction that $|G/Z(G)| \leq p^3$. As the nilpotency class of G is at least 3, its central quotient $G/Z(G)$ must therefore be nonabelian of order p^3. Since G is a 2-generated stem group, we thus have $|G/[G,G]| = p^2$. Furthermore, the derived subgroup of G is equal to $\langle [g_1,g_2],[g_1,g_2,g_1],[g_1,g_2,g_2]\rangle$ with $[g_1,g_2,g_1]$ and $[g_1,g_2,g_2]$ of order dividing p. We thus obtain the bound $|G| = |G/\gamma_2(G)| \cdot |\gamma_2(G)/\gamma_3(G)| \cdot |\gamma_3(G)| \leq p^5$, a contradiction. Applying the inequality $|G/Z(G)| \geq p^4$, the commuting probability bound and the information on the number of generators, the degree equation yields

$$(p+1)/p^4 < 1/py + 1/z. \tag{5.2}$$

Assuming $y \geq p^3$, we also have $z \geq p^3$, which is in conflict with inequality (5.2). Hence $y = p^2$, and inequality (5.2) additionally gives $z \leq p^3$. □

As in the case when $|G : \Phi(G)| = p^3$, the bound $z \leq p^3$ restricts the nilpotency class of G to at most 4. Note that the commutator $[g_2,g_1,g_2]$ is either trivial or equals a power of $[g_3,g_2]$ since $|[g_2,G]| = p^2$. We therefore have $\gamma_2(G)/\gamma_3(G) = \langle [g_2,g_1]\rangle$, $\gamma_3(G)/\gamma_4(G) = \langle [g_2,g_1,g_1]\rangle$, and $\gamma_4(G) = \langle [g_2,g_1,g_1,g_1]\rangle$ with all the groups being of order p. Moreover, both $g_1^{p^2}$ and $g_2^{p^2}$ are central in G as the Frattini subgroup is abelian. When $p=2$, this already gives $|G| \leq 2^{4+3} = 2^7$, a contradiction. Similarly, if the group G is of nilpotency class 3, we obtain $|G| \leq p^6$, another contradiction. The remaining case is dealt with in the following lemma.

Lemma. *If p is odd and G is of nilpotency class 4, then $|G| \leq p^6$.*

Proof. Note that we have $[g_2, g_1^p] = [g_3,g_2]^k$ for some k. This in turn gives

$$[g_2^p, g_1] = [g_2,g_1]^p [g_2,g_1,g_2]^{\binom{p}{2}} = [g_2,g_1]^p.$$

We also have $[g_2,g_1^p] = [g_2,g_1]^p[g,g_1]$ for some $g \in [G,G]$ that satisfies $[g_2,g] = 1$. Combining the two, we obtain $[g_2^p,g_1] = [g_2,g_1^p][g^{-1},g_1][g_2,g_1,g_2]^{\binom{p}{2}} = [g_1,f]^k[g^{-1},g_1] = [h,g_1]$ for some $h \in [G,G]$. Since $f \in \gamma_3(G)$, we also have $[g_2,h] = 1$. This shows that $g_2^p h^{-1} \in Z(G)$. As the group G is stem, we therefore have $|G/[G,G]| \leq p^3$. Hence $|G| = |G/[G,G]| \cdot |\gamma_2(G)| \leq p^6$. □

The proof of Theorem 5.7 is thus complete.

5.2.5 Applications

We will now go through some of the applications of Theorem 5.7. A nonprobabilistic criterion for the vanishing of the Bogomolov multiplier is first established.

Corollary 5.10. *Let G be a finite group. If $|[G,G]|$ is cubefree, then $\mathrm{B}_0(G)$ is trivial.*

Proof. Let S a nonabelian Sylow p-subgroup of G. By counting only the linear characters of S, we obtain the bound $\mathrm{k}(S) > |S : [S,S]| \geq |S|/p^2$, which further gives $\mathrm{cp}(S) > 1/p^2 \geq (2p^2+p-2)/p^5$. Theorem 5.7 implies $\mathrm{B}_0(S) = 0$. As the p-part of $\mathrm{B}_0(G)$ embeds into $\mathrm{B}_0(S)$, we conclude $\mathrm{B}_0(G) = 0$. □

The restriction to third powers of primes in Corollary 5.10 is best possible, as shown by the B_0-minimal groups given in Theorem 4.45, whose derived subgroups are of order p^3. We remark that another way of stating Corollary 5.10 is by saying that the Bogomolov multiplier of a finite extension of a group of cubefree order by an abelian group is trivial. This may be compared with [Bog87, Lemma 4.9].

We now apply Corollary 5.8 to provide some curious examples of B_0-minimal isoclinism families, determined by their stem groups. These in particular show that there is indeed no upper bound on the nilpotency class of a B_0-minimal group.

***Example* 5.11.** *For every $n \geq 6$, consider the group*

$$G_n = \left\langle \; a, b, c \; \middle| \; \begin{array}{l} a^2 = b^2 = 1, \; c^2 = [a, c], \\ [c, b] = [c, {}_{n-1} a], \; [b, a] \text{ central, class } n \end{array} \right\rangle .$$

Note that G can be viewed as a semidirect product of D_8 by $C_{2^{n-3}}$ with $D_8 = \langle a, b\rangle$, $C_{2^{n-3}} = \langle c\rangle$, and the action given by $c^a = c^{-1}$ and $b^a = c^{1+2^{n-4}}$.

Another way of presenting G_n is by a polycyclic generating sequence g_i, $1 \leq i \leq n$, subject to the following relations: $g_1^2 = g_2^2 = 1$, $g_i^2 = g_{i+1}g_{i+2}$ for $2 < i < n-2$, $g_{n-2}^2 = g_{n-1}$, $g_{n-1}^2 = g_n^2 = 1$, $[g_2, g_1] = g_n$, $[g_i, g_1] = g_{i+1}$ for $2 < i < n-1$, $[g_{n-1}, g_1] = [g_n, g_1] = 1$, $[g_3, g_2] = g_{n-1}$, and all the nonspecified commutators are trivial. Note that the group G_6 is the group given in Example 4.33. For any $n \geq 6$, the group G_n is a group of order 2^n and of nilpotency class $n-3$, generated by g_1, g_2, g_3. It is readily verified that $Z(G_n) = \langle g_{n-1}, g_n\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $[G_n, G_n] = \langle g_4, g_n\rangle \cong \mathbb{Z}/2^{n-4}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, whence G_n is a stem group. We claim that the group G_n is in fact a B_0-minimal group.

As we will be using Corollary 5.8, let us first inspect the conjugacy classes of G_n. It is straightforward that centralizers of noncentral elements of $\Phi(G_n)$ are all equal to the maximal subgroup $\langle g_2, g_3\rangle\Phi(G_n)$ of G_n. Furthermore, whenever the normal form of an element $g \in G_n \backslash \Phi(G_n)$ with respect to the above polycyclic generating sequence does not contain g_1, we have $C_{G_n}(g) = \langle g\rangle\Phi(G_n)$, and when the element g does have g_1 in its normal form, we have $C_{G_n}(g) = \langle g\rangle Z(G_n)$. Having determined the centralizers, we count the number of conjugacy classes in G_n. The central elements all form orbits of size 1. The elements belonging to $\Phi(G_n)\backslash Z(G_n)$ all have orbits of size $2^n/2^{n-1} = 2$ and there are $2^{n-3} - 4$ of them, which gives $2^{n-4} - 2$ conjugacy classes. Next, the elements not belonging to $\Phi(G_n)$ and not having g_1 in their normal form have orbits of size $2^n/2^{n-2} = 4$ and there are $3 \cdot 2^{n-3}$ of them, which gives $3 \cdot 2^{n-5}$ conjugacy classes. Finally, the elements that do have g_1 in their normal form each contribute one conjugacy class depending on the representative modulo $\Phi(G_n)$, which gives four conjugacy classes all together. Thus $\mathrm{k}(G_n) = 2^{n-4} + 3 \cdot 2^{n-5} + 6$, and hence $\mathrm{cp}(G_n) = 1/2^4 + 3/2^5 + 6/2^n$.

We now show that $\mathrm{B}_0(G_n) \cong \mathbb{Z}/2\mathbb{Z}$. First of all, we find the generator of $\mathrm{B}_0(G_n)$. The curly exterior square $G_n \curlywedge G_n$ is generated by the elements $g_3 \curlywedge g_2$ and $g_i \curlywedge g_1$ for $2 \leq i \leq n-2$. As G_n is metabelian, the group $G_n \curlywedge G_n$ is itself abelian. Any element $w \in G_n \curlywedge G_n$ may therefore be written in the form $w = (g_3 \curlywedge g_2)^\beta \prod_{i=2}^{n-2} (g_i \curlywedge g_1)^{\alpha_i}$ for some integers β, α_i. Note that w belongs to $\mathrm{B}_0(G_n)$ precisely when $[g_3, g_2]^\beta \prod_{i=2}^{n-2} [g_i, g_1]^{\alpha_i}$ is trivial. The latter product may be written in terms of the given polycyclic generating *sequence as* $\prod_{i=3}^{n-2} g_{i+1}^{\alpha_i} g_{n-1}^{\beta} g_n^{\alpha_2}$. *This implies* $\alpha_i = 0$ *for all* $2 \leq i < n-2$ *and* $\alpha_{n-2}+\beta \equiv 0$ *modulo* 2. *Note that we have* $(g_{n-2} \curlywedge g_1)^2 = g_{n-2} \curlywedge g_1^2 = 1$ *and similarly* $(g_3 \curlywedge g_2)^2 = 1$. *Denoting* $v = (g_3 \curlywedge g_2)(g_1 \curlywedge g_{n-2})^{-1}$, *we thus have* $\mathrm{B}_0(G_n) = \langle v \rangle$ *with* v *of order dividing* 2. *Let us now show that the element* v *is in fact nontrivial in* $G_n \curlywedge G_n$. *To this end, we construct a certain* B_0*-pairing* $\phi\colon G_n \times G_n \to \mathbb{Z}/2\mathbb{Z}$. *We define this pairing on tuples of elements of* G_n, *written in normal form. For* $g = \prod_{i=1}^{n} g_i^{a_i}$ *and* $h = \prod_{i=1}^{n} g_i^{b_i}$, *put*

$$\phi(g,h) = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + 2\mathbb{Z}.$$

We now show that ϕ *is indeed a* B_0*-pairing. It is straightforward that* ϕ *is bilinear and depends only on representatives modulo* $\Phi(G_n)$. *Suppose now that* $[x,y]=1$ *for some* $x,y \in G_n$. *If* $x \in \Phi(G_n)$, *then clearly* $\phi(x,y) = 2\mathbb{Z}$. *On the other hand, if* $x \notin \Phi(G_n)$, *then we must have* $y \in C_{G_n}(x) \leq \langle x \rangle \Phi(G_n)$ *by above, from which it follows that* $\phi(x,y) = \phi(x,x) = 2\mathbb{Z}$. *We have thus shown that the mapping* ϕ *is a* B_0*-pairing. Therefore* ϕ *determines a unique homomorphism of groups* $\phi^*\colon G_n \curlywedge G_n \to \mathbb{Z}/2\mathbb{Z}$ *such that* $\phi^*(g \curlywedge h) = \phi(g,h)$ *for all* $g,h \in G_n$. *As we have* $\phi^*(v) = \phi(g_3,g_2) - \phi(g_1,g_{n-2}) = 1+2\mathbb{Z}$, *the element* v *is nontrivial. Hence* $\mathrm{B}_0(G_n) = \langle v \rangle \cong \mathbb{Z}/2\mathbb{Z}$, *as required.*

The above determination of centralizers also enables us to show that every subgroup of G_n *has commuting probability greater than* $1/4$. *Note that it suffices to prove this only for maximal subgroups of* G_n. *To this end, let* M *be a maximal subgroup of* G_n. *Being of index* 2 *in* G_n, M *contains at least one of the elements* g_3, g_2, g_2g_3. *If it contains two of these, then we have* $M = \langle g_2, g_3 \rangle \Phi(G_n)$ *and so* $M/Z(M) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. *By [Gus73], this implies* $\mathrm{cp}(M) = 5/8$ *and we are done. Now assume that* M *contains exactly one of the elements* g_3, g_2, g_2g_3. *The centralizer of any element in* M *not belonging to* $\Phi(G_n)$ *is, by above, of index* $2^{n-1}/2^{n-2} = 2$ *in* M. *There are* $3 \cdot 2^{n-3}$ *of these elements, hence contributing* $3 \cdot 2^{n-4}$ *to the number of conjugacy classes in* M. *Similarly, the elements belonging to* $\Phi(G_n) \backslash Z(G_n)$ *all have their centralizer of index* $2^{n-1}/2^{n-2} = 2$ *in* M *and there are* $2^{n-3} - 4$ *of these elements, hence contributing* $2^{n-4} - 2$ *conjugacy classes in* M. *This gives* $\mathrm{k}(M) = 4 + 3 \cdot 2^{n-4} + (2^{n-4} - 2) > 2^{n-3}$ *and therefore* $\mathrm{cp}(M) > 1/4$. *It now follows from Corollary 5.8 that every proper subgroup of* G_n *has a trivial Bogomolov multiplier.*

Lastly, we verify that Bogomolov multipliers of proper quotients of G_n *are all trivial. To this end, let* N *be a proper normal subgroup of* G_n. *If* $g_{n-1} \in N$, *then the elements* g_2 *and* g_3 *commute in* G_n/N. *The group* $\langle g_2, g_3 \rangle \Phi(G_n) N$ *is therefore a maximal abelian subgroup of* G_n/N, *and it follows that* $\mathrm{B}_0(G_n/N) = 0$. *Suppose now that* $g_{n-1} \notin N$. *Note that we have* $G_n/N \simeq G_n/([G_n,G_n] \cap N)$ *by [Hal40]. Since the Bogomolov multiplier is an isoclinism invariant, we may assume that* N *is contained in* $[G_n,G_n] = \langle g_4, g_n \rangle \cong \mathbb{Z}/2^{n-4}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. *As* g_{n-1} *is the only element of order* 2 *in* $\langle g_4 \rangle$ *and* $g_{n-1} \notin N$, *we must have either* $N = \langle g_n \rangle$ *or* $N = \langle g_{n-1} g_n \rangle$. *Suppose first that* $N = \langle g_n \rangle$ *and consider the factor group* $H = G_n/N$. *Denoting* $v = (g_3 \curlywedge g_2)(g_1 \curlywedge g_{n-2})$, *we show as above that* $\mathrm{B}_0(H) = \langle v \rangle$. *Note that we have* $g_2 g_{n-2} \curlywedge g_1 g_3 = (g_2 \curlywedge g_3)(g_2 \curlywedge g_1)(g_{n-2} \curlywedge g_3)(g_{n-2} \curlywedge g_1) = v$ *in* H, *which implies that* v *is trivial in* $H \curlywedge H$, *whence* $\mathrm{B}_0(H) = 0$. *Now consider the case when* $N = \langle g_{n-1} g_n \rangle$ *and put* $H = G_n/N$. *Denoting* $v_1 = (g_3 \curlywedge g_2)(g_1 \curlywedge g_2)$ *and* $v_2 = (g_{n-2} \curlywedge g_1)(g_1 \curlywedge g_2)$, *we have* $\mathrm{B}_0(H) = \langle v_1, v_2 \rangle$. *Note* *that $g_1 \curlywedge g_2 g_{n-2} = v_1$ and $g_2 \curlywedge g_1 g_3 = v_2$ in H, which implies that v_1 and v_2 are both trivial in $H \curlywedge H$, whence* $\mathrm{B}_0(H) = 0$. *This completes the proof of the fact that G_n is a* B_0*-minimal group.*

We point out that these examples contradict a part of the statement of [Bog87, Theorem 4.6] and [Bog87, Lemma 5.4]. We found that the latter has been used in proving triviality of Bogomolov multipliers of finite almost simple groups [Kun10]. The claim is reduced to showing $\mathrm{B}_0(\mathrm{Out}(L))$ to be trivial for all finite simple groups L. Standard arguments from [Bog87] are then used to further reduce this to the case when L is of type $A_n(q)$ or $D_{2m+1}(q)$. These two cases are dealt with using the above erroneous claim. With some minor adjustments, the argument of [Kun10] can be saved as follows. First note that the linear groups $A_n(q)$ have been treated separately in [BMP04]. We remark that the argument for the exceptional case $n = 2, q = 9$ uses a consequence of the above statements, see also [HK11]. The result remains valid, since the Sylow 3-subgroup is abelian in this case. As for orthogonal groups $D_{2m+1}(q)$, note that the derived subgroup of $\mathrm{Out}(D_{2m+1}(q))$ is a subgroup of the group of outer-diagonal automorphisms, which is isomorphic to $\mathbb{Z}/(4, q-1)\mathbb{Z}$, see [Wil09]. Corollary 5.10 now gives the desired result.

Lastly, we say something about groups of small orders to which Theorem 5.7 may be applied. Given an odd prime p, it is readily verified using [Jam80] that among all isoclinism families of rank at most 5, only the family Φ_{10} has commuting probability lower or equal than $(2p^3+p-2)/p^5$. Theorem 5.7 therefore provides a unified explanation of the known result that Bogomolov multipliers of groups of order at most p^5 are trivial except for the groups belonging to the family Φ_{10} [HK11, Mor12 p5]. Next, consider the groups of order p^6. Out of a total of 43 isoclinism families, 19 of them have commuting probabilities exceeding the above bound. This includes all groups of nilpotency class 2. For 2-groups, use the classification [JNO90] to see that all commuting probabilities of groups of order at most 32 are all greater than 1/4. With groups of order 64, there are 237 groups with the same property out of a total of 267 groups. Again, this may be compared with known results [CHKK10]. Finally, one may use Theorem 5.7 on the Sylow subgroups of a given group rather than using Corollary 5.8 directly, thus potentially obtaining a better bound on commuting probability that ensures triviality of the Bogomolov multiplier.

5.3 Relative bounds

Let G be a group. The goal of this section is to provide relative bounds for $\mathrm{B}_0(G)$ in terms of $\mathrm{cp}(G)$.

5.3.1 CP extensions and commuting probability

It turns out that commutativity preserving extensions provide a natural setting for both commuting probability and Bogomolov multipliers. This is based on the following observation.

Proposition 5.12. *An extension* $N \longrightarrow G \xrightarrow{\pi} Q$ *is a central CP extension if and only if* $\mathrm{cp}(G) = \mathrm{cp}(Q)$.

Proof. Observe the homomorphism $\pi^2 \colon G \times G \to Q \times Q$. Note that commuting pairs in G map to commuting pairs in Q, hence

$$(\pi^2)^{-1}(\{(x, y) \in Q \times Q \mid [x, y] = 1\}) \supseteq \{(x, y) \in G \times G \mid [x, y] = 1\}. \tag{5.3}$$

The containment (5.3) is an equality if and only if the extension is CP and N is a central subgroup of G. On the other hand, notice that the fibres of π^2 are of order $|N|^2$, therefore $\mathrm{cp}(G) = |\{(x, y) \in G \times G \mid [x, y] = 1\}|/|G|^2 \leq |N|^2 |\{(x, y) \in Q \times Q \mid [x, y] = 1\}|/|G|^2 = \mathrm{cp}(Q)$ with equality precisely when (5.3) is an equality. This completes the proof. □

Consider a central extension $\langle z \rangle \longrightarrow G \xrightarrow{\pi} Q$. It follows from the above proof that this extension is a CP extension if and only if all conjugacy classes of Q lift with respect to π to exactly p different conjugacy classes in G.

The study of central CP extensions is thus equivalent to the study of extensions which preserve commuting probability. This may be exploited in providing a connection between the Bogomolov multiplier and commuting probability based on CP extensions. We give a simple example illustrating this.

Corollary 5.13. *For every number x in the range of the commuting probability function, there exists a group G with* $\mathrm{cp}(G) = x$ *and* $\mathrm{B}_0(G) = 0$.

Proof. Let Q be an arbitrary group with $\mathrm{cp}(Q) = x$, and let G be a CP-cover of Q. Then $\mathrm{cp}(G) = x$ by Proposition 5.12 and $\mathrm{B}_0(G) = 0$ by Theorem 4.19. □

Another way to look at this relation is on the level of isoclinism families. As a direct consequence of Corollary 4.21, we have that for every isoclinism family Φ and every subgroup N of $\mathrm{B}_0(\Phi)$, there is a family Φ' with $\mathrm{cp}(\Phi') = \mathrm{cp}(\Phi)$ and $\mathrm{B}_0(\Phi') = N$.

***Example* 5.14.** *Observe the isoclinism family Φ_{16} as given in Example 4.33. We have* $\mathrm{cp}(\Phi_{16}) = \mathrm{cp}(\Phi_{36}) = 1/4$, *while* $\mathrm{B}_0(\Phi_{16}) \cong C_2$ *and* $\mathrm{B}_0(\Phi_{36}) = 0$.

This connection also sheds new light on the results of the previous section. There, we have observed the structure of the Bogomolov multiplier while fixing a large commuting probability. Those results can be applied in the context of CP extensions.

Corollary 5.15. *Let Q be a finite group with* $\mathrm{cp}(Q) > 1/4$. *Then every central CP extension of Q is isoclinic to an extension with a trivial kernel.*

Proof. The Bogomolov multiplier of Q is trivial. Every central CP extension of Q is isoclinic to a stem extension by Lemma 4.15, and the kernel of the latter extension must be trivial by Theorem 4.16. □

5.3.2 Structural bounds

The theory of CP covers turns out to be of great use to bound the size of the Bogomolov multiplier in terms of the internal structure of the given group. The first result is an adaptation of the argument from [Jon74].

Proposition 5.16. *Let Q be a finite group and S a normal subgroup such that Q/S is cyclic. Then $|\mathrm{B}_0(Q)|$ divides $|\mathrm{B}_0(S)| \cdot |S^{\mathrm{ab}}|$, and $\mathrm{d}(\mathrm{B}_0(Q)) \leq \mathrm{d}(\mathrm{B}_0(S)) + \mathrm{d}(S^{\mathrm{ab}})$.*

Proof. Let G be a CP cover of Q. Thus G contains a subgroup $N \leq [G,G] \cap Z(G)$ such that $G/N \cong Q$ and $N \cong \mathrm{B}_0(Q)$. Choose X in G such that $X/N \cong S$. We may write $G = \langle u, X \rangle$ for some u. There is thus an epimorphism $\theta \colon X \to [G,G]/[X,X]$ given by $\theta(x) = [u,x][X,X]$. Therefore $|\mathrm{B}_0(Q)| = |N| = |N/(N \cap [X,X])| \cdot |N \cap [X,X]|$. Now, since $NX' \leq \ker\theta$, it follows that $|N/(N \cap [X,X])| \leq |[G,G]/[X,X]| \leq |X/N[X,X]| = |S^{\mathrm{ab}}|$. Observe that the CP covering extension G of Q induces a central CP extension X of S with kernel N. Whence by Lemma 4.15, we have that $N \cap [X,X]$ is the kernel of the associated stem extension. It now follows from Theorem 4.16 that $|N \cap [X,X]| \leq |\mathrm{B}_0(S)|$. This completes the proof of the first claim. For the second one, we similarly have $\mathrm{d}(\mathrm{B}_0(Q)) = \mathrm{d}(N) \leq \mathrm{d}(N/(N \cap [X,X])) + \mathrm{d}(N \cap [X,X])$. The result follows from $\mathrm{d}(N/(N \cap [X,X])) \leq \mathrm{d}([G,G]/[X,X]) \leq \mathrm{d}(S^{\mathrm{ab}})$. $\square$

Next, we also provide a bound for the exponent. This is an analogy of [JW73].

Proposition 5.17. *Let Q be a finite group and S a subgroup. Then $\mathrm{B}_0(Q)^{|Q:S|}$ embeds into $\mathrm{B}_0(S)$.*

Proof. Let G be a CP cover of Q. Again, G contains a subgroup $N \leq [G,G] \cap Z(G)$ such that $G/N \cong Q$ and $N \cong \mathrm{B}_0(Q)$. Choose X in G such that $X/N \cong S$. Consider the transfer map $\theta \colon G \to X/[X,X]$. Since N is central in G, we have $\theta(n) = n^{|Q:S|}[X,X]$ for all $n \in N$. But as $N \leq [G,G]$, we must also have that $N \leq \ker\theta$. Therefore $N^{|Q:S|} \leq N \cap [X,X]$. As in the proof of the previous proposition, we have that $N \cap [X,X]$ embeds into $\mathrm{B}_0(S)$. This completes the proof. $\square$

These results may be applied in various ways, depending on the structural properties of the group in question, to provide some absolute bounds on the order, rank or exponent of the Bogomolov multiplier. As an example, consider a p-group Q that has a maximal subgroup M with $\mathrm{B}_0(M) = 0$. The above propositions imply that for such groups, $\mathrm{B}_0(Q)$ is elementary abelian of rank at most $\mathrm{d}(M)$. Such groups include B_0-minimal groups. Since every B_0-minimal group can be generated by at most 4 elements, Schreier's index formula $\mathrm{d}(M) - 1 \leq |Q : M|(\mathrm{d}(Q) - 1)$, cf. [Rob96, 6.1.8], then gives an absolute upper bound on the number of generators of a maximal subgroup M. Whence the following corollary.

Corollary 5.18. *The Bogomolov multiplier of a B_0-minimal p-group is an elementary abelian group of rank at most $3p + 1$.*

Another direct application is to consider any abelian subgroup A of a given group Q. Since $\mathrm{B}_0(A) = 0$, we have the following.

Corollary 5.19. *Let Q be a finite group and A an abelian subgroup. Then* $\exp \mathrm{B}_0(Q)$ *divides* $|Q : A|$.

5.3.3 The ϵ bound

The bounds for the Bogomolov multiplier from the previous subsection can be applied in the setting of commuting probability. This may be thought of as a nonabsolute version of Corollary 5.15.

Theorem 5.20. *Let $\epsilon > 0$, and let Q be a group with* $\mathrm{cp}(Q) > \epsilon$. *Then* $|\mathrm{B}_0(Q)|$ *can be bounded in terms of a function of ϵ and* $\max\{\mathrm{d}(S) \mid S$ *a Sylow subgroup of* $Q\}$. *Moreover,* $\exp \mathrm{B}_0(Q)$ *can be bounded in terms of a function of ϵ.*

Proof. Since the p-part of $\mathrm{B}_0(Q)$ embeds into the Bogomolov multiplier of a p-Sylow subgroup of Q, we are immediately reduced to considering only p-groups. It follows from [Neu89, Ebe15] that Q has a subgroup K of nilpotency class 2 with $|Q : K|$ and $|[K, K]|$ both bounded by a function of ϵ. Applying Proposition 5.16 repeatedly on a sequence of subgroups from Q to K, each of index p in the previous one, it follows that $\mathrm{d}(\mathrm{B}_0(Q))$ can be bounded in terms of ϵ and $\mathrm{d}(\mathrm{B}_0(K))$. Now, $\mathrm{d}(\mathrm{B}_0(K)) \le \mathrm{d}(\mathrm{M}(K))$, and we can use the Ganea map $[K, K] \otimes K/[K, K] \to \mathrm{M}(K)$ whose cokernel embeds into $\mathrm{M}(K/[K, K])$. Note that $\mathrm{d}([K, K] \otimes K/[K, K]) \le \mathrm{d}(K)^2$ and $\mathrm{d}(\mathrm{M}(K/[K, K])) \le \binom{\mathrm{d}(K)}{2}$. Whence we obtain a bound for $\mathrm{d}(\mathrm{B}_0(Q))$ in terms of ϵ and $\mathrm{d}(Q)$. For the exponent, use Proposition 5.17 to bound $\exp \mathrm{B}_0(Q)$ by a function of $|Q : K|$ and $\exp \mathrm{B}_0(K)$. If K is abelian, then we are done. If not, then choose a commutator z in K. Set $J_z = \langle x \curlywedge y \mid [x, y] = z \rangle \le Q \curlywedge Q$, and denote by X the kernel of the map $\mathrm{B}_0(K) \to \mathrm{B}_0(K/\langle z \rangle)$. There is a commutative diagram as follows.

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & X \arrow[r] \arrow[d] & \mathrm{B}_0(K) \arrow[r] \arrow[d] & \mathrm{B}_0(K/\langle z \rangle) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & J_z \arrow[r] \arrow[d] & K \curlywedge K \arrow[r] \arrow[d] & K/\langle z \rangle \curlywedge K/\langle z \rangle \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \langle z \rangle \arrow[r] \arrow[d] & [K, K] \arrow[r] \arrow[d] & [K, K]/\langle z \rangle \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}
$$

Observe that $\exp J_z = p$, and so $\exp X = p$. It then follows that $\exp \mathrm{B}_0(K)$ is at most $p \cdot \exp \mathrm{B}_0(K/\langle z \rangle)$. Repeating this process with $K/\langle z \rangle$ instead of z until we reach an abelian group, we conclude that $\exp \mathrm{B}_0(K)$ divides $|[K,K]|$. The latter is bounded in terms of ϵ alone. The proof is now complete. $\square$

We record an intriguing corollary concerning the exponent of the Schur multiplier.

Corollary 5.21. *Given $\epsilon > 0$, there exists a constant $C = C(\epsilon)$ such that for every group Q with $\mathrm{cp}(Q) > \epsilon$, we have $\exp \mathrm{M}(Q) \leq C \cdot \exp Q$.*

Proof. We have that $\exp \mathrm{M}(Q) \leq \exp \mathrm{B}_0(Q) \cdot \exp \mathrm{M}_0(Q)$ and $\exp \mathrm{M}_0(Q) \leq \exp Q$. Now apply Theorem 5.20. $\square$

Finally, let us end by asking whether or not the rank of the group G in the Theorem 5.20 can be removed from the statement. Based on the proof, it suffices to consider only some special groups.

Question 5.22. *Let G be a p-group of nilpotency class 2 and $|[G,G]|$ bounded by an absolute constant. Does this imply that the rank of $\mathrm{B}_0(G)$ is also bounded?*

6

Rationality revisited

A new apparition of the Bogomolov multiplier is exposed. We get there by widening our context to Lie groups and consider their representations. A classical method of obtaining these is the Kirillov orbit method. We present the idea behind the method and give a more detailed description for the class of algebra groups. On the most basic level, this leads to the so called fake degree conjecture. We tackle it by considering algebra groups that arise from modular group rings. It is here that the Bogomolov multiplier enters into play via K-theoretical considerations, thus refuting the conjecture. Finally, we look at the situation from the point of view of algebraic groups and show how elements of the Bogomolov multiplier can be seen as rational points on a certain commutator variety.

This chapter is based on [GRJZJ, Oli80].

6.1 Kirillov orbit method

Let G be a Lie group. In an attempt to determine all irreducible unitary representations of G, Kirillov [Kir04] developed the *orbit method*. It says that an irreducible representation of G should roughly correspond to a symplectic manifold X equipped with a G-action.

The orbit method is based on the concept of *quantization* in mathematical physics. In a classical mechanical system, the phase space forms a symplectic manifold. Classical observables are functions on the phase space. In the presence of a symmetry G of the system, one may consider an orbit of G on the phase space to obtain what is called a homogeneous Poisson G-manifold. This may be regarded as a classical mechanical system equipped with a symmetry group G. It turns out that every such manifold is a cover of an orbit of G acting on the dual of its Lie algebra. On the other hand, the mathematical model for quantum mechanics is a Hilbert space consisting of wave functions. In a quantum mechanical system, the observables are self-adjoint operators on that space. In the presence of a symmetry G of the system, one may consider an orbit of G on the Hilbert space to obtain an irreducible unitary representation of G. This may be regarded as a quantum mechanical system equipped with a symmetry group G. Quantization says that to each classical mechanical system one can associate a corresponding quantum mechanical system. The orbit method is a version of this claim in the presence of a suitable group action.

6.1.1 The method

Let G be a group and $\mathfrak{g}$ a Lie algebra with a G-action. Imitating the classical situation of a Lie group and its Lie algebra, suppose there are *exponential- and logarithm-like* maps $e\colon \mathfrak{g} \to G$ and $l\colon G \to \mathfrak{g}$ that are compatible with the G-action on $\mathfrak{g}$ and the conjugation action of G on itself. There is an induced G-action on the dual space $\mathfrak{g}^* = \hom(\mathfrak{g}, \mathbb{C}^\times)$, called the *coadjoint action.* Consider the set of orbits $\mathcal{O}$ of this action. There is now a complicated quantization scheme applicable to general curved manifolds. In its simplified and most basic form, it is based on taking an orbit $\Omega \in \mathcal{O}$ and associating to it the average value of elements of Ω on a group element,

$$\chi_\Omega\colon G \to \mathbb{C}, \quad g \mapsto \frac{1}{\sqrt{|\Omega|}} \sum_{\lambda \in \Omega} \lambda(l(g)).$$

For certain groups G, Lie algebras $\mathfrak{g}$ and orbits Ω, the sum in the definition makes sense. In this way, we have produced a class function χ_Ω on G. It may be that this class function is in fact an irreducible character of G. Then, we have a map from $\mathcal{O}$ to $\mathrm{Irr}(G)$, mapping Ω to χ_Ω. If this whole procedure works and the final map $\mathcal{O} \to \mathrm{Irr}(G)$ is a bijection, we say that Kirillov's orbit method is valid in the given setting.

***Example* 6.1.** *Take G to be a nilpotent connected simply connected Lie group over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$. Let $\mathfrak{g}$ be its Lie algebra equipped with a natural G-action. In this setting, the Kirillov orbit method works and gives a bijection between the coadjoint orbits of G on $\mathfrak{g}^*$ and* $\mathrm{Irr}(G)$*.*

***Example* 6.2.** *Take $G = \mathrm{SL}_2(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ with the natural G-action. In this setting, the Kirillov orbit method is not valid. There are exceptional representations that are unnoticed by the quantization procedure.*

6.1.2 Algebra groups

Let A be a nilpotent finite dimensional $\mathbb{F}_q$-algebra with q a p-power for some prime p. Take $G = 1 + A$ to be the set $\{1 + a \mid a \in A\}$ equipped with the operation $(1 + a)(1 + b) = 1 + a + b + ab$. Then G is a finite group. Such groups that arise from nilpotent algebras are called *algebra groups.* A prototype of an algebra group is the group $\mathrm{UT}_n(q)$. Representation theory of algebra groups is rich, difficult and of interest. Since there is a natural correspondence between elements of G and elements of A, many Lie- theoretic techniques can be used to deal with algebra groups. Kazhdan [Kaz77] showed that even the potent Kirillov's orbit method can be applied under some assumptions. The setup is as follows. Define an action of $1 + b \in G$ on $a \in A$ by $a^{1+b} = (1 + b)^{-1}a(1 + b)$. There is an induced action of $1 + b \in G$ on $\lambda \in A^*$ by $\lambda^{1+b}(a) = \lambda(a^{(1+b)^{-1}})$. Assuming $A^p = 0$, the maps

$$\exp\colon A \to 1 + A,\ a \mapsto \sum_{i=0}^{\infty} \frac{a^i}{i!}, \quad \log\colon 1 + A \to A,\ 1 + a \mapsto \sum_{i=1}^{\infty} (-1)^{i+1} \frac{a^i}{i}$$

are well-defined and G-equivariant. Using these, Kazhdan was able to prove that the average of a G-orbit in A^* is a character that is in fact induced from a linear algebra subgroup of G.

Theorem 6.3 ([Kaz77]). *Let A be a nilpotent finite dimensional $\mathbb{F}_q$-algebra with $A^p = 0$. Let $G = 1 + A$. Let Ω be a G-orbit in A^* and take $\lambda \in \Omega$. Then there is a subalgebra $C \leq A$ and $\psi \in \mathrm{Lin}(1+C)$ such that $|A : C| = \sqrt{|\Omega|}$, $\psi(1+c) = \lambda(c)$ for all $c \in C$, and $\psi \uparrow^G = \frac{1}{\sqrt{|\Omega|}} \sum_{\lambda' \in \Omega} \lambda'(\log(1+c))$.*

One can then obtain an explicit expression that gives a bijective correspondence between the characters of G and the orbits of A^*, see [San01]. There are obstructions to extending Kazhdan's result to all algebra groups. Some consequences of his result can, however, be extended to an arbitrary algebra group with no restriction on the nilpotency class. Isaacs [Isa95] has proved that irreducible characters of an $\mathbb{F}_q$-algebra group G all have q-power degree. Later, Halasi [Hal04] showed that every irreducible character of G is induced from a linear character of an algebra subgroup $H \leq G$. Thus, there are positive results for both character degrees and linear induction.

6.1.3 Fake degree conjecture

It was conjectured by M. Isaacs that the Kirillov orbit method should work for arbitrary algebra groups solely on the level of degrees. Take an orbit Ω and its associated function χ_Ω. Note that we have $\chi_\Omega(1) = \sqrt{|\Omega|}$. The values $\sqrt{|\Omega|}$ with Ω varying are called *fake degrees.*

Conjecture 6.4 (Fake degree conjecture). *In every algebra group the character degrees coincide, counting multiplicities, with the fake degrees.*

Focusing only on linear characters, the fake degree conjecture would establish a bijection between linear characters of G and fixed points of A^* under the coadjoint action of G. Take $g = 1 + u$ to be an element of G and let $\lambda \in A^*$. Then g fixes λ if and only if for every $v \in A$, $\lambda(gvg^{-1}) = \lambda(v)$ or equivalently $\lambda(gvg^{-1} - v) = 0$. Since multiplication by g acts bijectively on A this amounts to $\lambda(gv - vg) = \lambda(uv - vu) = \lambda([u,v]_L) = 0$ for every $v \in A$. Thus, λ is a fixed point if and only if $\lambda([A,A]_L) = 0$. The number of fixed points in A^* therefore equals the number of linear forms vanishing on $[A,A]_L$. This implies the following corollary of the fake degree conjecture.

Lemma 6.5. *Let $G = 1 + A$ be an algebra group. If the fake degree conjecture holds for G, then $|A/[A,A]_L| = |(1+A)_{\mathrm{ab}}|$.*

Thus in order to understand Conjecture 6.4, one should first answer the following question.

Question 6.6. *Is it true that the size of the group abelianization of $1 + A$ coincides with the size of the Lie abelianization of A?*

In [Jai04] an example that provides a negative answer to Question 6.6 in characteristic 2 was constructed. However, in questions related to character correspondences for finite p-groups the prime $p = 2$ always plays a special role (see, for example, [Jai06]), and so one might hope that Conjecture 6.4 still holds in odd characteristic. We will show how the Bogomolov multiplier enters the game to give a negative answer.

6.2 Modular group rings

Let X be a finite p-group. Given a ring R we will set I_R to be the augmentation ideal of the group ring $R[X]$. If we take $R = \mathbb{F}_q$, then $\mathrm{I}_{\mathbb{F}_q}$ is a nilpotent algebra and $1 + \mathrm{I}_{\mathbb{F}_q}$ is the group of normalized units of the modular group ring $\mathbb{F}_q[X]$.

6.2.1 Lie abelianization

It is easy to compute the Lie abelianization in the case of group rings.

Lemma 6.7. *Let X be a finite group and $\mathbb{F}$ a field. Then*

$$\dim_{\mathbb{F}} \mathrm{I}_{\mathbb{F}} / [\mathrm{I}_{\mathbb{F}}, \mathrm{I}_{\mathbb{F}}]_L = \mathrm{k}(X) - 1.$$

Proof. It is clear that the set X is an $\mathbb{F}$-basis for $\mathbb{F}[X]$. We first claim that

$$\dim_{\mathbb{F}} \mathbb{F}[X]/[\mathbb{F}[X], \mathbb{F}[X]]_L = \mathrm{k}(X).$$

Let $x_1, \dots, x_{\mathrm{k}(X)}$ be representatives of conjugacy classes of X. Observe that for any $x, y, g \in X$ with $y = g^{-1}xg$, we have $x - y = [g, g^{-1}x]_L$. The elements $\bar{x}_1, \dots, \bar{x}_{\mathrm{k}(X)}$ therefore span $\mathbb{F}[X]/[\mathbb{F}[X], \mathbb{F}[X]]_L$.

Set λ_i to be the linear functional on $\mathbb{F}[X]$ that takes the value 1 on the elements corresponding to the conjugacy class of x_i and vanishes elsewhere. Observe that for any $g, h \in X$, we have $[g, h]_L = g(hg)g^{-1} - hg$ and hence each λ_i induces a linear functional on $\mathbb{F}[X]/[\mathbb{F}[X], \mathbb{F}[X]]_L$. Now if $\sum_j \alpha_j \bar{x}_j = 0$ for some $\alpha_j \in \mathbb{F}$, then $\alpha_i = \lambda_i(\sum_j \alpha_j \bar{x}_j) = 0$ for each i. It follows that $\bar{x}_1, \dots, \bar{x}_{\mathrm{k}(X)}$ are also linearly independent and hence a basis. This proves the claim.

Now, it is clear that $\{g - 1 : g \in X \setminus \{1\}\}$ is an $\mathbb{F}$-basis for $\mathrm{I}_{\mathbb{F}}$. Since for any $g, h \in X$, we have $[g, h]_L = [g - 1, h - 1]_L$, it follows that $[\mathbb{F}[X], \mathbb{F}[X]]_L = [\mathrm{I}_{\mathbb{F}}, \mathrm{I}_{\mathbb{F}}]_L$, whence the lemma. □

6.2.2 Reduced Whitehead groups

It is substantially more difficult to compute the size of the group abelianization $|(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}|$. In order to achieve this, we first lift the problem to zero characteristic using K-theoretical methods.

Given a ring R, recall the first K-theoretical group $\mathrm{K}_1(\mathrm{R}) = \mathrm{GL}(\mathrm{R})_{\mathrm{ab}}$. When R is a local ring, there is an isomorphism $\mathrm{K}_1(\mathrm{R}) \cong \mathrm{R}^*_{\mathrm{ab}}$ (see [Ros94, Corollary 2.2.6]). We therefore have

$$\mathrm{K}_1(\mathbb{F}_q[X]) \cong \mathbb{F}_q^* \times (1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}.$$

Let ζ_n denote a primitive n-root of unity. If p is a prime and q is a power of p, let $\mathrm{R}_q = \mathbb{Z}_p[\zeta_{q-1}]$ be a finite extension of the p-adic integers $\mathbb{Z}_p$. Note that $\mathrm{R}_q / p\,\mathrm{R}_q \cong \mathbb{F}_q$. Our proof relies on inspecting the connection between $\mathrm{K}_1(\mathbb{F}_q[X])$ and $\mathrm{K}_1(\mathrm{R}_q[X])$, the advantage being that the latter group has been studied in detail by Oliver [Oli80].

Put Q_q to be the ring of fractions of R_q and let

$$\mathrm{SK}_1(\mathrm{R}_q[X]) = \ker\Big(\mathrm{K}_1(\mathrm{R}_q[X]) \to \mathrm{K}_1(\mathrm{Q}_q[X])\Big).$$

The group $\mathrm{SK}_1(\mathrm{R}_q[X])$ is the *reduced Whitehead group*. It is, in a sense, a measure of the failure of the determinant map over the ring $\mathrm{R}_q[X]$ versus $\mathrm{Q}_q[X]$. Denote

$$\mathrm{Wh}'(\mathrm{R}_q[X]) = \mathrm{K}_1(\mathrm{R}_q[X])/(\mathrm{R}_q^* \times X_{\mathrm{ab}} \times \mathrm{SK}_1(\mathrm{R}_q[X])).$$

The group $\mathrm{Wh}'(\mathrm{R}_q[X])$ is torsion-free (cf. [Wal74]), giving an explicit description

$$\mathrm{K}_1(\mathrm{R}_q[X]) \cong \mathrm{R}_q^* \times \mathrm{SK}_1(\mathrm{R}_q[X]) \times X_{\mathrm{ab}} \times \mathrm{Wh}'(\mathrm{R}_q[X]). \tag{6.1}$$

To transfer this description to the finite case of $\mathrm{K}_1(\mathbb{F}_q[X])$, we need to worry about understanding torsion. This is precisely what the main result of [Oli80] does.

Fix a $\mathbb{Z}_p$-basis $B_q = \{\lambda_j \mid 1 \le j \le n\}$ of R_q and let φ be a generator of $\mathrm{Aut}(\mathrm{R}_q | \mathbb{Z}_p) \cong \mathrm{Gal}(\mathbb{F}_q|\mathbb{F}_p)$ such that $\varphi(\lambda) \cong \lambda^p \pmod p$. Let us define

$$\bar{\mathrm{I}}_{\mathrm{R}_q} = \mathrm{I}_{\mathrm{R}_q} / \langle x - x^g \mid x \in \mathrm{I}_{\mathrm{R}_q},\, g \in X\rangle.$$

Set $\mathcal{C}$ to be a set of nontrivial conjugacy class representatives of X. Then $\bar{\mathrm{I}}_{\mathrm{R}_q}$ can be regarded as a free $\mathbb{Z}_p$-module with basis $\{\lambda\overline{(1-r)} \mid \lambda \in B_q,\, r \in \mathcal{C}\}$. The crux of understanding the structure of the group $\mathrm{K}_1(\mathrm{R}_q[X])$ is in the following short exact sequence.

Theorem 6.8 ([Oli80], Theorem 2)**.** *There is a short exact sequence*

$$1 \longrightarrow \mathrm{Wh}'(\mathrm{R}_q[X]) \xrightarrow{\Gamma} \bar{\mathrm{I}}_{\mathrm{R}_q} \xrightarrow{\omega} X_{\mathrm{ab}} \longrightarrow 1.$$

Main idea. The map Γ is defined by composing the p-adic logarithm with a linear automorphism of $\bar{\mathrm{I}}_{\mathrm{R}_q} \otimes \mathbb{Q}_p$. More precisely, there is a map $\mathrm{Log}\colon 1 + \mathrm{I}_{\mathrm{R}_q} \to \mathrm{I}_{\mathrm{R}_q} \otimes \mathbb{Q}_p$, which induces an injection $\log\colon \mathrm{Wh}'(\mathrm{R}_q[X]) \to \bar{\mathrm{I}}_{\mathrm{R}_q} \otimes \mathbb{Q}_p$. Setting $\Phi\colon \mathrm{I}_{\mathrm{R}_q} \to \mathrm{I}_{\mathrm{R}_q}$ to be the map $\sum_{g\in X} \alpha_g g \mapsto \sum_{g \in X} \varphi(\alpha_g) g^p$, we define $\Gamma\colon \mathrm{Wh}'(\mathrm{R}_q[X]) \to \bar{\mathrm{I}}_{\mathrm{R}_q} \otimes \mathbb{Q}_p$ as the composite of log followed by the linear map $1 - \frac{1}{p}\Phi$. It is shown in [Oli80, Proposition 10] that $\operatorname{im}\Gamma \subseteq \bar{\mathrm{I}}_{\mathrm{R}_q}$, i.e., Γ is integer-valued. The situation can be recapped with the diagram

$$\begin{tikzcd} & 1 + \mathrm{I}_{\mathrm{R}_q} \arrow[d] \arrow[dr, bend left, "(1-\frac{1}{p}\Phi)\circ\log"] & \\ 1 \arrow[r] & \mathrm{Wh}'(\mathrm{R}_q[X]) \arrow[r, "\Gamma"] & \bar{\mathrm{I}}_{\mathrm{R}_q}. \end{tikzcd} \tag{6.2}$$

The map ω is defined by $\omega(\sum_i \lambda_i x_i) = \prod_i x_i^{\mathrm{tr}(\lambda_i)}$. $\square$

Utilizing the above theorem, the main result of [Oli80] goes as follows.

Theorem 6.9 ([Oli80], Theorem 3)**.** *There is a natural isomorphism* $\mathrm{SK}_1(\mathrm{R}_q[X]) \cong \mathrm{B}_0(X)$.

Main idea. We show that the objects $\mathrm{SK}_1(\mathrm{R}[X])$ and $\mathrm{B}_0(X)$ behave in the same way when producing group extensions of X. Going all the way to the CP cover of X then gives the desired isomorphism. To do so, first take an arbitrary extension of p-groups

$$1 \longrightarrow Y \xrightarrow{\iota} \tilde{X} \xrightarrow{\pi} X \longrightarrow 1.$$

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{SK}_1(\mathrm{R}_q[Y]) \arrow[r] \arrow[d] & \mathrm{Wh}(\mathrm{R}_q[Y]) \arrow[r, "\Gamma"] \arrow[d] & \bar{\mathrm{I}}_{\mathrm{R}_q}(Y) \arrow[r, "\omega"] \arrow[d, "\bar{\mathrm{I}}(\iota)"] & Y_{\mathrm{ab}} \arrow[r] \arrow[d, "\iota_{\mathrm{ab}}"] & 0 \\
0 \arrow[r] & \mathrm{SK}_1(\mathrm{R}_q[\tilde{X}]) \arrow[r] \arrow[d, "\pi_*"] & \mathrm{Wh}(\mathrm{R}_q[\tilde{X}]) \arrow[r, "\Gamma"] \arrow[d] & \bar{\mathrm{I}}_{\mathrm{R}_q}(\tilde{X}) \arrow[r, "\omega"] \arrow[d, "\mathrm{Wh}(\pi)"] & Y_{\mathrm{ab}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{SK}_1(\mathrm{R}_q[X]) \arrow[r] & \mathrm{Wh}(\mathrm{R}_q[X]) \arrow[r, "\Gamma"] & \bar{\mathrm{I}}_{\mathrm{R}_q}(X) \arrow[r, "\omega"] & Y_{\mathrm{ab}} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Column-wise composites are trivial. There is a boundary map

$$\Delta = \mathrm{Wh}(\pi) \circ \Gamma^{-1} \circ \bar{\mathrm{I}}(\iota) \circ \omega^{-1} \colon \ker(\iota_{\mathrm{ab}}) \to \operatorname{coker}(\pi_*).$$

Let κ_π be the composite

$$\kappa_\pi \colon Y \cap [\tilde{X}, \tilde{X}] \to \ker(\iota_{\mathrm{ab}}) \to \operatorname{coker}(\pi_*).$$

Then κ_π factors through the subgroup $Y \cap \mathrm{K}(\tilde{X})$. It is shown in [Oli80, Proposition 16] that the induced map

$$\kappa_\pi \colon \frac{Y \cap [\tilde{X}, \tilde{X}]}{Y \cap \mathrm{K}(\tilde{X})} \to \operatorname{coker}(\pi_*)$$

is an isomorphism. By Hopf's formula, there is a natural epimorphism

$$\delta_\pi \colon \mathrm{B}_0(X) \to \frac{[\tilde{X}, \tilde{X}]}{Y \cap \mathrm{K}(\tilde{X})}.$$

Replacing the original extension by a maximal stem CP extension, we can assume that δ_π is an isomorphism. It is verified in [Oli80, Proposition 18] that the composite map

$$\mathrm{SK}_1(\mathrm{R}_q[X]) \to \operatorname{coker}(\pi_*) \leftarrow \frac{[\tilde{X}, \tilde{X}]}{Y \cap \mathrm{K}(\tilde{X})} \leftarrow \mathrm{B}_0(X)$$

is then also an isomorphism. $\square$

6.2.3 Group abelianization

Finally we can compute the size of the abelianization $(1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$ by projecting the result from the previous section.

Theorem 6.10. *Let X be a finite p-group. Then $|(1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}| = q^{\mathrm{k}(X)-1}|\mathrm{B}_0(X)|$.*

Before embarking on the proof, we emphasize that as there are plenty of finite p-groups with non-trivial Bogomolov multipliers, we obtain a negative solution to the fake degree conjecture for all primes.

Corollary 6.11. *For every prime p there exists a finite dimensional nilpotent $\mathbb{F}_p$-algebra A such that the size of the abelianization of $1+A$ is greater than the index of $[A,A]_L$ in A. In particular, the fake degree conjecture is not valid in any characteristic.*

Define the abelian group M_q to be

$$\mathrm{M}_q = \bar{\mathrm{I}}_{\mathrm{R}_q} / \langle p\lambda\overline{(1-r)} - \varphi(\lambda)\overline{(1-r^p)} \mid \lambda \in B_q,\ r \in \mathcal{C}\rangle.$$

The proof of Theorem 6.10 rests on the following more informative structural description of the group $(1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$.

Theorem 6.12. *Let X be a finite p-group. There is an exact sequence*

$$1 \longrightarrow \mathrm{B}_0(X)\times X_{\mathrm{ab}} \longrightarrow (1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}} \longrightarrow \mathrm{M}_q \longrightarrow X_{\mathrm{ab}} \longrightarrow 1.$$

Proof. To relate the above results to $\mathrm{K}_1(\mathbb{F}_q[X])$, we invoke a part of the K-theoretical long exact sequence for the ring $\mathrm{R}_q[X]$ with respect to the ideal generated by p,

$$\mathrm{K}_1(\mathrm{R}_q[X],p) \xrightarrow{\partial} \mathrm{K}_1(\mathrm{R}_q[X]) \xrightarrow{\mu} \mathrm{K}_1(\mathbb{F}_q[X]) \longrightarrow 1. \tag{6.3}$$

Note that $\mathrm{K}_1(\mathrm{R}_q[X],p) = (1+p\,\mathrm{R}_q)\times \mathrm{K}_1(\mathrm{R}_q[X],p\,\mathrm{I}_{\mathrm{R}_q})$ and $\mathrm{R}_q^*/(1+p\,\mathrm{R}_q)\cong \mathbb{F}_q^*$. Hence (6.1) and (6.3) give a reduced exact sequence

$$\mathrm{K}_1(\mathrm{R}_q[X],p\,\mathrm{I}_{\mathrm{R}_q}) \xrightarrow{\partial} \mathrm{Wh}'(\mathrm{R}_q[X]) \xrightarrow{\mu} \frac{(1+\mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}}{\mu(\mathrm{SK}_1(\mathrm{R}_q[X])\times X_{\mathrm{ab}})} \longrightarrow 1. \tag{6.4}$$

To determine the structure of the relative group $\mathrm{K}_1(\mathrm{R}_q[X],p\,\mathrm{I}_{\mathrm{R}_q})$ and its connection to the map ∂, we make use of [Oli80, Proposition 2]. The restriction of the logarithm map Log to $1+p\,\mathrm{I}_{\mathrm{R}_q}$ induces an isomorphism log: $\mathrm{K}_1(\mathrm{R}_q[X],p\,\mathrm{I}_{\mathrm{R}_q}) \to p\bar{\mathrm{I}}_{\mathrm{R}_q}$ such that the following diagram commutes:

$$\begin{tikzcd} & 1+p\,\mathrm{I}_{\mathrm{R}_q} \arrow[d] \arrow[dr, "\log"] & \\ 1 \arrow[r] & \mathrm{K}_1(\mathrm{R}_q[X],p\,\mathrm{I}_{\mathrm{R}_q}) \arrow[r, "\log"] & p\bar{\mathrm{I}}_{\mathrm{R}_q}. \end{tikzcd} \tag{6.5}$$

In particular, the group $\mathrm{K}_1(\mathrm{R}_q[X],p\,\mathrm{I}_{\mathrm{R}_q})$ is torsion-free, and so $\mu(\mathrm{SK}_1(\mathrm{R}_q[X])\times X_{\mathrm{ab}}) \cong \mathrm{SK}_1(\mathrm{R}_q[X])\times X_{\mathrm{ab}}$. Note that by [Bas68, Theorem V.9.1], the vertical map $1+p\,\mathrm{I}_{\mathrm{R}_q} \to \mathrm{K}_1(\mathrm{R}_q[X],p\,\mathrm{I}_{\mathrm{R}_q})$ of the above diagram is surjective.

We now collect the stated results to prove the theorem. First combine the diagrams (6.2) and (6.5) into the following diagram:

$$
\begin{tikzcd}
1 + p\,\mathrm{I}_{\mathrm{R}_q} \arrow[rr] \arrow[d] \arrow[dr, "\log"] & & 1 + \mathrm{I}_{\mathrm{R}_q} \arrow[d] \arrow[dr, "(1-\frac{1}{p}\Phi)\circ\log"] & \\
\mathrm{K}_1(\mathrm{R}_q[X], p\,\mathrm{I}_{\mathrm{R}_q}) \arrow[rr, "\partial"] \arrow[dr, "\log"'] & & \mathrm{Wh}'(\mathrm{R}_q[X]) \arrow[dr, "\Gamma"] & \\
& p\bar{\mathrm{I}}_{\mathrm{R}_q} \arrow[rr, "1-\frac{1}{p}\Phi"] & & \bar{\mathrm{I}}_{\mathrm{R}_q}.
\end{tikzcd}
\tag{6.6}
$$

Since the back and top rectangles commute and the left-most vertical map is surjective, it follows that the bottom rectangle also commutes. Whence $\operatorname{coker}\partial \cong \operatorname{coker}(1 - \frac{1}{p}\Phi)$. Observing that the latter group is isomorphic to M_q, the exact sequence (6.4) gives an exact sequence

$$1 \longrightarrow \mathrm{B}_0(X) \times X_{\mathrm{ab}} \longrightarrow (1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}} \longrightarrow \mathrm{M}_q \longrightarrow X_{\mathrm{ab}} \longrightarrow 1. \tag{6.7}$$

The proof is complete. □

We now derive Theorem 6.10 from Theorem 6.12.

Proof of Theorem 6.10. It suffices to compute $|\mathrm{M}_q|$. To this end, we filter M_q by the series of its subgroups

$$\mathrm{M}_q \supseteq p\,\mathrm{M}_q \supseteq p^2\,\mathrm{M}_q \supseteq \cdots \supseteq p^{\log_p(\exp X)}\,\mathrm{M}_q\,.$$

Note that the relations $p\lambda\overline{(1-r)} - \varphi(\lambda)\overline{(1-r^p)} = 0$ imply $p^{\log_p(\exp X)}\,\mathrm{M}_q = 0$.

For each $0 \leq i \leq \log_p(\exp X)$, put

$$X_i = \{x^{p^i} \mid x \in X\} \text{ and } \ \mathcal{C}_i = \mathcal{C} \cap (X_i \setminus X_{i+1}).$$

Then

$$p^i\,\mathrm{M}_q\,/p^{i+1}\,\mathrm{M}_q = \langle \lambda\overline{(1-r)} : \lambda \in B_q,\ r \in \mathcal{C}_i\rangle \cong \bigoplus_{\mathcal{C}_i} C_p^n.$$

It follows that $|\mathrm{M}_q| = q^{|\mathcal{C}|} = q^{\mathrm{k}(X)-1}$ and the proof is complete. □

***Example* 6.13.** *Let X be the group given in Section 4.3.2, Family 39. It is a group of order* 128 *with $X_{\mathrm{ab}} \cong C_4 \times C_4$. Its Bogomolov multiplier is generated by the commutator relation $[g_3, g_2] = [g_5, g_1]$ of order* 2. *We have $\mathrm{k}(X) = 26$ and by inspecting the power structure of conjugacy classes, we see that $\mathrm{M}_q \cong C_2^{13} \times C_4^6$. On the other hand, using the available computational tool [LAGUNA], it is readily verified that we have $(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}} \cong C_2^{13} \times C_4^5 \times C_8$. Following the proof of Theorem 6.12, the embedding of* $\mathrm{B}_0(X) \times X_{\mathrm{ab}}$ *into* $(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$ *maps the generating relation* $[g_3, g_2] = [g_5, g_1]$ *of* $\mathrm{B}_0(X)$ *into the element* $\exp((1 - g_7)(g_3 - g_5))$*, which belongs to $(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}^4$. In particular, the embedding of* $\mathrm{B}_0(X) \times X_{\mathrm{ab}}$ *into* $(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$ *may not be split.*

6.3 Rationality of commutators

In this final section we provide a conceptual explanation for the equality in Theorem 6.10. Instead of lifting the original problem to zero characteristic, we algebraically close the field.

6.3.1 Algebraic groups

Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$. One can think of $\mathbf{G} = 1 + \mathrm{I}_{\mathbb{F}}$ as an algebraic group defined over $\mathbb{F}_p$. It is clear that $\mathbf{G}$ is a unipotent group. A direct calculation shows that the Lie algebra $\mathfrak{L}(\mathbf{G})$ of $\mathbf{G}$ is isomorphic to $\mathrm{I}_{\mathbb{F}}$. Consider the finite case as being embedded in the algebraically closed case. We write $\mathbf{G}(\mathbb{F}_q)$ for the $\mathbb{F}_q$-points of $\mathbf{G}$. The derived subgroup $\mathbf{G}'$ of $\mathbf{G}$ is also a unipotent algebraic group defined over $\mathbb{F}_p$ (see [Bor91, Corollary I.2.3]), and so by [KMT74, Remark A.3], we have $|\mathbf{G}'(\mathbb{F}_q)| = q^{\dim \mathbf{G}'}$. Note that in general we have only an inclusion

$$(1 + \mathrm{I}_{\mathbb{F}_q})' = (\mathbf{G}(\mathbb{F}_q))' \subseteq \mathbf{G}'(\mathbb{F}_q),$$

but not the equality.

6.3.2 Relations as rational points

There seem to be more questions to answer in the case of an algebraically closed field, and they are interrelated. As in the finite case, the connecting tissue is the Bogomolov multiplier. We determine the dimensions of $\mathbf{G}'$ and of $[\mathfrak{L}(\mathbf{G}), \mathfrak{L}(\mathbf{G})]$, and show how the irregularity from the finite case can be seen here.

Theorem 6.14. *Let X be a finite p-group and $\mathbf{G} = 1 + \mathrm{I}_{\mathbb{F}}$.*

1. *We have*
$$\dim \mathbf{G}' = \dim_{\mathbb{F}} [\mathfrak{L}(\mathbf{G}), \mathfrak{L}(\mathbf{G})]_L = |X| - \mathrm{k}(X).$$
In particular,
$$|\mathbf{G}(\mathbb{F}_q) : \mathbf{G}'(\mathbb{F}_q)| = q^{\mathrm{k}(X)-1}.$$

2. *For every $q = p^n$, we have*
$$\mathbf{G}'(\mathbb{F}_q)/\mathbf{G}(\mathbb{F}_q)' \cong \mathrm{B}_0(X).$$

The second statement of the above theorem offers a new interpretation of the Bogomolov multiplier. As such, it promises new ways of understanding its structure. We give an example of this reasoning.

Theorem 6.15. *Let X be a finite p-group and $\mathbf{G} = 1 + \mathrm{I}_{\mathbb{F}}$. For every $q = p^n$, we have*

$$\exp \mathrm{B}_0(X) = \min\{m \mid \mathbf{G}'(\mathbb{F}_q) \subseteq \mathbf{G}(\mathbb{F}_{q^m})'\}.$$

The relevance of Theorem 6.15 comes from a classical problem concerning the Schur multiplier. One would like to understand what is the relation between the exponent of a finite group and of its Schur multiplier ([Sch04]). It is known that the exponent of the Schur multiplier is bounded by some function that depends only on the exponent of the group ([Mor07]), but this bound is obtained from the bounds that appear in the solution of the Restricted Burnside Problem and so it is probably very far from being optimal. The exponent of the Schur multiplier is at most the product of the exponent of the group by the exponent of the Bogomolov multiplier. Thus, we hope that Theorem 6.15 would help obtain a better bound on the exponent of the Schur multiplier.

Proof of Theorem 6.14 and 6.15. We will consider an extension $\mathbb{F}_l$ of $\mathbb{F}_q$ of degree m. The inclusion $\mathbf{G}(\mathbb{F}_q) \subseteq \mathbf{G}(\mathbb{F}_l)$ induces a map $f\colon \mathbf{G}(\mathbb{F}_q)_{\mathrm{ab}} \to \mathbf{G}(\mathbb{F}_l)_{\mathrm{ab}}$ with $\ker f = (\mathbf{G}(\mathbb{F}_q) \cap \mathbf{G}(\mathbb{F}_l)')/\mathbf{G}(\mathbb{F}_q)'$ Note that there exists a large enough m such that $\mathbf{G}'(\mathbb{F}_q) = \mathbf{G}(\mathbb{F}_q) \cap \mathbf{G}(\mathbb{F}_l)'$, and hence $\ker f = \mathbf{G}'(\mathbb{F}_q)/\mathbf{G}(\mathbb{F}_q)'$. For this reason we want to understand $\ker f$ for a given m.

The inclusion $\mathbb{F}_q \subseteq \mathbb{F}_l$ induces a map $\mathrm{incl}\colon \mathrm{K}_1(\mathbb{F}_q[X]) \to \mathrm{K}_1(\mathbb{F}_l[X])$. Note that f is just the restriction of incl to $(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$. Recalling sequence (6.4) from the proof of Theorem 6.12, we set

$$\mathrm{SK}_1(\mathbb{F}_l[X]) = \mu(\mathrm{SK}_1(\mathrm{R}_l[X]) \subseteq (1 + \mathrm{I}_{\mathbb{F}_l[X]})_{\mathrm{ab}} = \mathbf{G}(\mathbb{F}_l)/\mathbf{G}(\mathbb{F}_l)'.$$

Commutativity of the diagram

$$\begin{tikzcd} \mathrm{K}_1(\mathrm{R}_q[X]) \arrow[r, "\mu"] \arrow[d, "\mathrm{incl}"] & \mathrm{K}_1(\mathbb{F}_q[X]) \arrow[d, "\mathrm{incl}"] \\ \mathrm{K}_1(\mathrm{R}_l[X]) \arrow[r, "\mu"] & \mathrm{K}_1(\mathbb{F}_l[X]) \end{tikzcd} \tag{6.8}$$

shows that incl restricts to a map $\mathrm{incl}\colon \mathrm{SK}_1(\mathbb{F}_q[X]) \to \mathrm{SK}_1(\mathbb{F}_l[X])$. Recalling that $\mathrm{SK}_1(\mathrm{R}_l[X]) \cong \mathrm{B}_0(X)$, we obtain from sequence (6.7) the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{SK}_1(\mathbb{F}_q[X]) \times X_{\mathrm{ab}} \arrow[r] \arrow[d, "\mathrm{incl} \times \mathrm{id}"] & \mathbf{G}(\mathbb{F}_q)_{\mathrm{ab}} \arrow[r] \arrow[d, "f"] & \mathrm{M}_q \arrow[r] \arrow[d, "\iota"] & X_{\mathrm{ab}} \arrow[r] & 1 \\ 1 \arrow[r] & \mathrm{SK}_1(\mathbb{F}_l[X]) \times X_{\mathrm{ab}} \arrow[r] & \mathbf{G}(\mathbb{F}_l)_{\mathrm{ab}} \arrow[r] & \mathrm{M}_l \arrow[r] & X_{\mathrm{ab}} \arrow[r] & 1, \end{tikzcd}$$

where ι is the map induced by the inclusion $\mathrm{I}_{\mathrm{R}_q} \subseteq \mathrm{I}_{\mathrm{R}_l}$.

We will now show that $\ker \iota = 0$. This will imply $\ker f \subseteq \mathrm{SK}_1(\mathbb{F}_q[X])$. Without loss of generality, we may assume that there is an inclusion of bases $B_q \subseteq B_l$. As in the proof of Theorem 6.10, let us consider the series

$$\mathrm{M}_l \supseteq p\,\mathrm{M}_l \supseteq p^2\,\mathrm{M}_l \supseteq \ldots \supseteq p^{\log_p(\exp X)}\,\mathrm{M}_l\,.$$

Observe again that for each $0 \leq i \leq \exp X - 1$ we have

$$p^i\,\mathrm{M}_q\,/p^{i+1}\,\mathrm{M}_q = \langle \lambda\overline{(1-r)} : \lambda \in B_q,\ r \in \mathcal{C}_i \rangle,$$
$$p^i\,\mathrm{M}_l\,/p^{i+1}\,\mathrm{M}_l = \langle \lambda\overline{(1-r)} : \lambda \in B_l,\ r \in \mathcal{C}_i \rangle.$$

If we consider the graded groups associated to the series above, we get an induced map

$$\mathrm{gr}(\iota)\colon \bigoplus_{i=0}^{\exp X-1} p^i\,\mathrm{M}_q\,/p^{i+1}\,\mathrm{M}_q \to \bigoplus_{i=0}^{\exp X-1} p^i\,\mathrm{M}_l\,/p^{i+1}\,\mathrm{M}_l\,.$$

By construction ι is induced by the assignments $\lambda\overline{(1-r_m)} \mapsto \lambda\overline{(1-r_m)}$, for every $\lambda \in B_q,\ r \in \mathcal{C}$. Hence $\mathrm{gr}(\iota)$ is injective in every component and therefore injective. This implies $\ker\iota = 0$, as desired. In particular, we obtain that

$$|\mathbf{G}(\mathbb{F}_q)/\mathbf{G}'(\mathbb{F}_q)| \geq |\,\mathrm{M}_q\,| = q^{\mathrm{k}(X)-1}. \tag{6.9}$$

We are now ready to show the first statement of Theorem 6.14. Observe that $\mathbf{G}$ is a unipotent connected algebraic group defined over $\mathbb{F}_p$ and so is $\mathbf{G}'$ ([Bor91, Corollary I.2.3]). Hence $\mathbf{G}' \cong_{\mathbb{F}_p} \mathbb{A}^{\dim\mathbf{G}'}$ (c.f. [KMT74, Remark A.3]) and so $|\mathbf{G}'(\mathbb{F}_p)| = p^{\dim\mathbf{G}'}$. By (6.9), we have $|\mathbf{G}(\mathbb{F}_p)/\mathbf{G}'(\mathbb{F}_p)| \geq p^{\mathrm{k}(X)-1}$, whence $\dim\mathbf{G}' \leq |X| - \mathrm{k}(X)$. On the other hand we have $[\mathfrak{L}(G),\mathfrak{L}(G)]_L = [\mathrm{I}_{\mathbb{F}},\mathrm{I}_{\mathbb{F}}]_L$, which, by Lemma 6.7, has dimension $|X| - \mathrm{k}(X)$. It is well known that for an algebraic group, $\dim\mathbf{G}' \geq \dim[\mathfrak{L}(\mathbf{G}),\mathfrak{L}(\mathbf{G})]_L$ (see [Hum75, Corollary 10.5]). Thus $\dim\mathbf{G}' = |X| - \mathrm{k}(X)$.

Let us set $e = \exp\mathrm{B}_0(X)$. We now claim that $\ker f = \mathrm{SK}_1(\mathbb{F}_q[X])$ if and only if e divides $m = |\mathbb{F}_l : \mathbb{F}_q|$. This will imply the second statement of Theorem 6.14 and also Theorem 6.15.

Consider $\mathbb{F}_l[X] \cong \bigoplus_{i=1}^m \mathbb{F}_q[X]$ as a free $\mathbb{F}_q[X]$-module. This gives a natural inclusion $\mathrm{GL}_1(\mathbb{F}_l[X]) \to \mathrm{GL}_m(\mathbb{F}_q[X])$, which induces the transfer map

$$\mathrm{trf} : \mathrm{K}_1(\mathbb{F}_l[X]) \to \mathrm{K}_1(\mathbb{F}_q[X]).$$

Note that if $x \in \mathrm{K}_1(\mathbb{F}_q[X])$, then $(\mathrm{trf}\circ\mathrm{incl})(x) = x^m$. By commutativity of (6.8) the transfer map restricts to a map $\mathrm{trf} : \mathrm{SK}_1(\mathbb{F}_l[X]) \to \mathrm{SK}_1(\mathbb{F}_q[X])$. Moreover, by [Oli80, Proposition 21] the transfer map is an isomorphism. It thus follows that $\mathrm{incl}(\mathrm{SK}_1(\mathbb{F}_q[X])) = 1$ if and only if e divides m. Hence $\ker f = \mathrm{SK}_1(\mathbb{F}_q[X])$ if and only if e divides m and we are done. □

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Razširjeni povzetek

Pričujoča disertacija se tiče komutatorjev, bolj ali manj v abstraktnih grupah. Vzamemo grupo G, ponavadi končno, in dva njena elementa $x, y \in G$. Njun *komutator* je grupni element $[x, y] = x^{-1}y^{-1}xy$. Predstavljamo si ga kot merilec nekomutativnosti elementov x in y. Ker je $[x, y]$ grupni element, lahko z grupno operacijo primerjamo različne načine nekomutativnosti v grupi G. Ob danem naboru komutatorjev pravimo, da obstaja *relacija* med temi komutatorji, če se nek netrivialen produkt teh komutatorjev izračuna v enoto grupe G.

Nekatere komutatorske relacije so zgolj posledica algebraičnih manipulacij. Takšna je, na primer, relacija $[x, y][y, x] = 1$ v grupi G, ki ne prinaša nobene vsebine v zvezi s strukturo grupe G. Takšnim relacijam in njihovim posledicam pravimo *univerzalne komutatorske relacije*. Izhajajo iz komutatorskih relacij prostih grup in se jih da hitro dobro razumeti. V delu se zato osredotočimo na potencialen obstoj komutatorskih relacij, ki niso univerzalne. Poleg tega v raziskovanju izločimo vpliv enostavnih komutatorjev, ki se v grupi G izračunajo v enoto. Na ta način odstranimo trivialne prispevke v komutatorskih relacijah in med njimi izoliramo tiste zares netrivialne. Vse te netrivialne neuniverzalne komutatorske relacije je mogoče zbrati v abelovo grupo, ki jo imenujemo *multiplikator Bogomolova* grupe G, označen z $\mathrm{B}_0(G)$.

Multiplikator Bogomolova je objekt osrednjega interesa disertacije. V tem kontekstu ga je nedavno vpeljal Moravec [Mor12], sloneč na delu Millerja [Mil52] s pogledom proti interpretaciji Bogomolova nerazvejenih Brauerjevih grup kvocientnih raznoterosti [Bog87]. Multiplikator Bogomolova predstavlja obstrukcijo; njegova trivialnost pomeni natanko to, da vse netrivialne komutatorske relacije grupe G sledijo iz univerzalnih komutatorskih relacij, če le dojemamo relacije iz komutirajočih parov kot trivialne. Posebej nas torej zanima, ali je grupa $\mathrm{B}_0(G)$ trivialna. Imeti želimo tudi nekakšen nadzor nad njenim obnašanjem. V disertaciji predstavimo razne vidike tega. Ekspozicija sloni na [FAJ, GRJZJ, Jez14, JM14 GAP, JM14 128, JM15, JM].

Univerzalne komutatorske relacije

Delo pričnemo z osnovnimi definicijami in predstavitvijo motivacije. V Razdelku 2.1 formalno vpeljemo komutatorske relacije. Ob dani grupi G opazujmo prosto grupo $(G, G) = \langle (x, y) \mid x, y \in G \rangle$, generirano z množico $G \times G$. Na ta način vsakemu komutatorju v grupi G priredimo simbol v prosti grupi (G, G). Obstaja naraven epimorfizem $\kappa\colon (G, G) \to G$ z lastnostjo $\kappa(x, y) = [x, y]$ za vse $x, y \in G$. *Relacija* med komutatorji

$$[x_1, y_1],\ [x_2, y_2],\ \ldots,\ [x_n, y_n]$$

grupe G je beseda $\omega \in (G, G)$ z nosilcem v množici $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$, za katero velja $\kappa(w) = 1$ in G. Grupo vseh komutatorskih relacij označimo z $\mathfrak{R}_G = \ker \kappa$. Grupa G deluje s konjugiranjem po komponentah na grupi $\mathfrak{R}_G$. Za neka elementa $\omega_1, \omega_2 \in (G, G)$ bomo oznako $\omega_1 = \omega_2$ interpretirali v grupi $(G, G)/\mathfrak{R}_G$, kar pomeni, da je $\omega_1 \omega_2^{-1}$ relacija.

Med vsemi komutatorskimi relacijami izpostavimo univerzalne. V ta namen naj bo F prosta grupa skupaj z nekim homomorfizmom $\varphi\colon F \to G$. Tedaj obstaja induciran homomorfizem $\varphi_{\mathfrak{R}}\colon \mathfrak{R}_F \to \mathfrak{R}_G$. Na ta način lahko prenesemo relacije proste grupe $\mathfrak{R}_F$ v grupo relacije grupe G. Relacijam $\mathfrak{R}_F$ zato pravimo *univerzalne komutatorske relacije*. Pri tem imamo v mislih poljubno prosto grupo F. Ob specifični grupi G uporabimo isto ime za unijo slih vseh možnih homomorfizmov $\varphi_{\mathfrak{R}}$ za poljubno prosto grupo F. Millerjev izrek nam da enostavno generirajočo množico univerzalnih komutatorskih relacij.

Izrek (glej Theorem 2.9)**.** *Naj bo F prosta grupa. Tedaj lahko grupo $\mathfrak{R}_F$ generiramo kot podgrupo edinko grupe (F, F) z relacijami*

$$(xy, z) = (x, z)^y (y, z), \quad (x, yz) = (x, z)(x, y)^z, \quad (x, x) = 1 \tag{6.10}$$

za vse $x, y, z \in F$.

V razdelku 2.2 sestavimo ambientni objekt, preko katerega izražamo neuniverzalnost komutatorskih relacij. Imenuje se *vnanji kvadrat* grupe G. To je grupa $G \wedge G$, generirana s simboli $x \wedge y$ za $x, y \in G$ glede na naslednje relacije:

$$xy \wedge z = (x^y \wedge z^y)(y \wedge z), \quad x \wedge yz = (x \wedge z)(x^z \wedge y^z), \quad x \wedge x = 1 \tag{6.11}$$

za $x, y, z \in G$. Zopet vidimo epimorfizem $\kappa\colon G \wedge G \to [G, G]$ definiran z $x \wedge y \mapsto [x, y]$. Njegovo jedro $\mathrm{M}(G) = \ker \kappa$ je torej grupa komutatorskih relacij grupe G, ki niso posledice univerzalnih relacij. Znano je, da je grupa $\mathrm{M}(G)$ izomorfna homološki grupi $\mathrm{H}_2(G, \mathbb{Z})$.

Da odstranimo trivialne prispevke iz komutirajočih parov v neuniverzalnih relacijah, konstrukcijo vnanjega kvadrata ustrezno faktoriziramo. Naj bo $\mathrm{M}_0(G) = \langle x \wedge y \mid x, y \in G,\ [x, y] = 1 \rangle$, to je grupa trivialnih komutatorskih relacij med vsemi neuniverzalnimi. *Kodrasti vnanji kvadrat* grupe G je grupa $G \curlywedge G = (G \wedge G)/\,\mathrm{M}_0(G)$. Zopet vidimo epimorfizem $\kappa\colon G \curlywedge G \to [G, G]$ z lastnostjo $x \curlywedge y \mapsto [x, y]$ za $x, y \in G$. Nazadnje dospemo do grupe netrivialnih neuniverzalnih relacij

$$\mathrm{B}_0(G) = \ker \kappa = \mathrm{M}(G)/\,\mathrm{M}_0(G),$$

to je *multiplikator Bogomolova.*

Kratka motivacija za študij tega objekta je predstavljena v Razdelku 2.3. Sloni na pristopu Noetherjeve k iskanju protiprimerov za problem racionalnosti v algebraični geometriji. Tu najdemo izvor imenovanja multiplikatorja Bogomolova.

Osnovne lastnosti in primeri

V razdelku 3.1 raziščemo nekaj osnovnih lastnosti multiplikatorja Bogomolova. Najprej pokažemo, kako lahko Hopfovo homološko formulo prilagodimo za multiplikator Bogomolova. Tako dobimo opis objekta $\mathrm{B}_0(G)$ v odvisnosti od neke proste prezentacije dane grupe G. V nekaterih primerih lahko s to formulo eksplicitno določimo izomorfnostni tip grupe $\mathrm{B}_0(G)$. V spodnji formuli je $\mathrm{K}(F)$ množica vseh komutatorjev v grupi F.

Izrek (glej Proposition 3.1)**.** *Naj bo grupa G dana s prosto prezentacijo $G = F/R$. Tedaj je*

$$\mathrm{B}_0(G) \cong \frac{[F,F] \cap R}{\langle \mathrm{K}(F) \cap R \rangle}.$$

Pokažemo tudi, da se objekta $G \curlywedge G$ in $\mathrm{B}_0(G)$ dobro ujameta s pojmom izoklinizma. Izoklinizem je ekvivalenčna relacija na množici vseh grup, ki združuje grupe glede na njihovo komutatorsko strukturo.

Izrek (glej Theorem 3.3)**.** *Naj bosta G in H izoklini grupi. Tedaj je* $\mathrm{B}_0(G) \cong \mathrm{B}_0(H)$.

Nadalje pokažemo, da je funktor B_0 multiplikativen.

Izrek (glej Theorem 3.5)**.** *Naj bosta G in H grupi. Tedaj je* $\mathrm{B}_0(G \times H) \cong \mathrm{B}_0(G) \times \mathrm{B}_0(H)$.

Raziskujemo tudi obnašanje multiplikatorja Bogomolova v povezavi s podgrupami in kvocienti dane grupe. V obeh primerih funktor B_0 ni nujno eksakten. Tolažilni rezultat za podgrupe je naslednji.

Izrek (glej Theorem 3.11)**.** *Naj bo G končna grupa in P p-podgrupa Sylowa grupe G. Tedaj je slika inducirane preslikave $\mathrm{B}_0(P) \to \mathrm{B}_0(G)$ enaka p-podgrupi Sylowa $\mathrm{B}_0(G)_p$ grupe $\mathrm{B}_0(G)$. Še več, $\mathrm{B}_0(G)_p$ je izomorfna direktnemu sumandu grupe $\mathrm{B}_0(P)$.*

Pri kvocientih lažje razumemo defekt.

Izrek (glej Theorem 3.6)**.** *Naj bo G grupa in N podgrupa edinka grupe G. Tedaj obstaja eksaktno zaporedje*

$$\mathrm{B}_0(G) \longrightarrow \mathrm{B}_0(G/N) \longrightarrow \frac{N}{\langle \mathrm{K}(G) \cap N \rangle} \longrightarrow G^{\mathrm{ab}} \longrightarrow (G/N)^{\mathrm{ab}} \longrightarrow 0.$$

Po obravnavanju teh osnovnih lastnosti v razdelku 3.2 podamo mnogo primerov grup s trivialnimi in netrivialnimi multiplikatorji Bogomolova. Pri tem prikažemo razne tehnike dokazovanja. Najprej obravnavamo grupe z velikimi abelovimi podgrupami.

Izrek (glej Theorem 3.13)**.** *Naj bo G končna grupa in A njena abelova podgrupa edinka z lastnostjo, da je grupa G/A ciklična. Tedaj je* $\mathrm{B}_0(G) = 0$.

Nato si ogledamo simetrične grupe, končne enostavne grupe, Burnsideove grupe in grupe enično zgornje trikotnih matrik nad končnimi polji. Dlje se zadržimo pri p-grupah malih moči in malih razredov nilpotentnosti. Študiranje multiplikatorjev Bogomolova za p-grupe razreda nilpotentnosti 2 in eksponenta p (predpostavimo $p > 2$) se prevede na problem linearne algebre nad končnimi polji. Ob takšni grupi G ranga d je namreč struktura grupe G natanko določena z množico svojih komutatorskih relacij. Na te lahko gledamo kot na elemente vektorskega prostora $\mathbb{F}_p^d \wedge \mathbb{F}_p^d$ prek korespondence $[x, y] \equiv x \wedge y$. Izbrana baza $\{z_i \mid 1 \leq i \leq d\}$ prostora $\mathbb{F}_p^d$ nam da bazo $\{z_i \wedge z_j \mid 1 \leq i < j \leq d\}$ prostora $\mathbb{F}_p^d \wedge \mathbb{F}_p^d$. Množica komutatorskih relacij grupe G tako tvori nek linearen podprostor R prostora $\mathbb{F}_p^d \wedge \mathbb{F}_p^d$. V tem smislu komutirajoči pari grupe G ustrezajo elementarnim klinom v R, to je elementom oblike $x \wedge y$ za neka $x, y \in \mathbb{F}_p$. Te elementi vektorskega prostora $\mathbb{F}_p^d \wedge \mathbb{F}_p^d$ lahko opišemo kot točke na algebraični raznoterosti $\mathfrak{P}$, določeni s Plückerjevimi relacijami. V primeru, ko velja $d = 4$, imamo eno samo relacijo

$$Z_{12}Z_{34} + Z_{13}Z_{42} + Z_{14}Z_{23} = 0,$$

kjer koordinata Z_{ij} v prostoru $\mathbb{F}_p^4 \wedge \mathbb{F}_p^4$ predstavlja koordinato vektorja $z_i \wedge z_j$. Grupa G je tako dana z izbiro podprosotra R v 6-dimenzionalnem vektorskem prostoru $\mathbb{F}_p^4 \wedge \mathbb{F}_p^4$. Njen multiplikator Bogomolova prepoznamo kot

$$\mathrm{B}_0(G) \cong \frac{R}{\langle \mathfrak{P} \cap R \rangle}.$$

Določiti moramo le še presek $\mathfrak{P} \cap R$. To storimo tako, da parametriziramo elemente R z ustrezno bazo in tako izračunamo kvadratno formo, ki jo določa restrikcija $\mathfrak{P}$ na R. Vprašanje trivialnosti $\mathrm{B}_0(G)$ se tu izraža kot vprašanje o tem, ali rešitve kvadratne forme $\mathfrak{P}|_R$ razpenjajo ves prostor R.

Nazadnje si ogledamo še p-grupe maksimalnega razreda nilpotentnosti. Ta primer je posebej zahteven in prav tu prvič najdemo naravne primere grup, ki posedujejo mnogo netrivialnih neuniverzalnih relacij. Ta rezultat potem omogoči konstrukcijo grup poljubnega korazreda z velikimi multiplikatorji Bogomolova. Korazred grupe moči p^n in razreda nilpotentnosti c je število $\bar{c} = n - c$.

Izrek (glej Corollary 3.27 in Theorem 3.31)**.** *Za vsako praštevilo p in celi števili $\bar{c} \geq 1$ ($\bar{c} \geq 2$ za $p = 2$) ter $C > 0$ obstaja neskončno mnogo p-grup G korazreda $\bar{c}$ z $|\mathrm{B}_0(G)| > C$.*

Razklenjanje relacij

Multiplikator Bogomolova ima homološko in v dualu tudi kohomološko interpretacijo. Komutatorske relacije si lahko zatorej predstavljamo kot posebne razširitve grup. Teorijo teh razširitev pričnemo razvijati v Razdelku 4.1; karakterizirane so kot razširitve, ki ohranjajo komutativnost. Natančneje, razširitev grupe N z grupo Q je kratno eksaktno zaporedje

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1 .$$

Rečemo, da ta razširitev *ohranja komutativnost* (krajše, je *OK-razširitev*), če imajo komutirajoči pari v Q komutirajoče dvige v G preko epimorfizma π. Omejimo se na razširitve z abelovim jedrom N. Ob dani grupi Q in Q-modulu N zberemo vse OK-razširitve grupe N s Q v kohomološki objekt $\mathrm{H}^2_{\mathrm{CP}}(Q, N)$. Ta kot podgrupa običajne kohomološke grupe $\mathrm{H}^2(Q,N)$ sestoji iz tistih kociklov $\omega \in \mathrm{Z}^2(Q,N)$, za katere velja, da za vsak komutirajoči par $x_1, x_2 \in Q$ obstajata $a_1, a_2 \in N$, da velja

$$\omega(x_1,x_2) - \omega(x_2,x_1) = \partial_{a_1}(x_1) + \partial_{a_2}(x_2).$$

Pokažemo, da ta kohomološka grupa parametrizira OK-razširitve.

Izrek (glej Proposition 4.4)**.** *Naj bo Q grupa in N njen modul. Ekvivalenčni razredi OK-razširitev grupe N s Q so v bijektivni korespondenci z elementi grupe* $\mathrm{H}^2_{\mathrm{CP}}(Q,N)$*.*

Nadalje se osredotočimo na centralne razširitve. To so tiste, pri katerih je jedro N vsebovano v centru razširitve G. Multiplikator Bogomolova vstopi kot univerzalni objekt, ki parametrizira vse centralne razširitve dane grupe. Dokažemo različico izreka o univerzalnih koeficientih.

Izrek (glej Theorem 4.8)**.** *Naj bo N trivialen Q-modul. Tedaj obstaja razcepno eksaktno zaporedje*

$$0 \longrightarrow \mathrm{Ext}(Q^{\mathrm{ab}},N) \longrightarrow \mathrm{H}^2_{\mathrm{CP}}(Q,N) \longrightarrow \mathrm{Hom}(\mathrm{B}_0(Q),N) \longrightarrow 0\,.$$

Izpeljemo nekaj alternativnih karakterizacij OK-razširitev. Te slonijo na lastnostih jedra razširitve.

Izrek (glej Proposition 4.10)**.** *Centralna razširitev*

$$1 \longrightarrow N \xrightarrow{\ \chi\ } G \xrightarrow{\ \pi\ } Q \longrightarrow 1$$

je OK-razširitev natanko tedaj, ko velja $\chi(N) \cap \mathrm{K}(G) = 1$*.*

Pokažemo tudi, da se OK-razširitve dobro ujamejo s pojmom *izoklinizma* razširitev. Grobo rečeno sta dve razširitvi izoklini, če se ujemata na nivoju komutatorskih podgrup. Na ta način lahko povežemo razširitve, ki opisujejo enako dogajanje na nivoju komutatorjev. Ekvivalenčne razrede lahko opišemo z delovanjem na multiplikatorju Bogomolova.

Izrek (glej Theorem 4.14)**.** *Izoklini razredi centralnih OK-razširitev grupe Q so v bijektivni korespondenci z orbitami delovanja grupe* $\mathrm{Aut}\,Q$ *na podgrupah grupe* $\mathrm{B}_0(Q)$*.*

Z Razdelkom 4.2 pričnemo razvijati teorijo krovnih grup. Najprej pokažemo, da se lahko omejimo zgolj na razširitve grupe N s Q, za katere je jedro razširitve N vsebovano v $Z(Q) \cap [Q,Q]$. Takim razširitvam pravimo *zarodne* centralne razširitve. Ob dani grupi Q je njen *krov, ki ohranja komutativnost* (krajše, *OK-krov*) takšna zarodna centralna raširitev, ki ohranja komutativnost in katere jedro je moči $|\mathrm{B}_0(Q)|$. Pokažemo, da OK-krovi posedujejo krovno lastnost.

Izrek (glej Theorem 4.16)**.** *Naj bo Q končna grupa, dana s prosto prezentacijo $Q = F/R$. Postavimo*

$$H = \frac{F}{\langle \mathrm{K}(F) \cap R\rangle} \quad \textit{and} \quad A = \frac{R}{\langle \mathrm{K}(F) \cap R\rangle}.$$

1. *Grupa A je končno generirana centralna podgrupa grupe H. Njena torzijska podgrupa je*
$$T(A) = \frac{[F,F] \cap R}{\langle K(F) \cap R \rangle} \cong \mathrm{B}_0(Q).$$

2. *Naj bo C komplement torzije $T(A)$ v A. Tedaj je H/C OK-krov grupe Q. Na ta način dobimo vsak OK-krov grupe Q.*

3. *Naj bo G zarodna centralna OK-razširitev grupe N s Q. Tedaj je G homomorfna slika grupe H. V posebnem je N homomorfna slika $\mathrm{B}_0(Q)$. Torej, OK-krovi grupe Q so zarodne centralne OK-razširitve grupe G maksimalne moči.*

Pokažemo, da je multiplikator Bogomolova OK-krova trivialen. Na ta način je mogoče videti komutatorske relacije kot zanke v topološkem prostoru, ki jih s prehodom na ustrezen krov lahko razklenemo.

Izrek (glej Corollary 4.21)**.** *Naj bo Q grupa in G njen OK-krov. Vsaki filtraciji podgrup*

$$1 = N_0 \leq N_1 \leq \cdots \leq N_n = \mathrm{B}_0(Q)$$

lahko pridružimo zaporedje grup $G_i = G/N_i$, kjer je G_i centralna OK-razširitev grupe G_j z jedrom $N_j/N_i \cong \mathrm{B}_0(G_j)/\mathrm{B}_0(G_i)$ za $i \leq j$.

S tem lahko bolje razumemo tako maksimalne kot minimalne OK-razširitve.

Izrek (glej Theorem 4.30)**.** *Grupa $\mathrm{H}^2_{\mathrm{CP}}(Q, \mathbb{Z}/p\mathbb{Z})$ je elementarno abelova ranga $\mathrm{d}(Q) + \mathrm{d}(\mathrm{B}_0(Q))$.*

Konstrukcije OK-krovov uporabimo v Razdelku 4.3, kjer predstavimo učinkovit algoritem za računanje multiplikatorjev Bogomolova končnih rešljivih grup. Algoritem sloni na računanju s policikličnimi prezentacijami in na koncu prepozna komutatorske relacije dane grupe, ki generirajo njen multiplikator Bogomolova. Poženemo ga na vseh grupah moči 128 in predstavimo rezultate.

V Razdelku 4.4 raziskujemo grupe, ki so minimalne glede na posedovanje neuniverzalnih komutatorskih relacij. Te grupe so osnovni sestavni deli grup z netrivialnimi multiplikatorji Bogomolova. Končni grupi G rečemo B_0-*minimalna grupa*, kadar je $\mathrm{B}_0(G) \neq 0$, za vsako njeno pravo podgrupo H in vsako njeno pravo podgrupo edinko N pa velja $\mathrm{B}_0(H) = \mathrm{B}_0(G/N) = 0$. Najdemo nekaj strukturnih omejitev takih grup, najpomembnejša je naslednja.

Izrek (glej Theorem 4.40)**.** *Naj bo G B_0-minimalna grupa. Tedaj je G končna p-grupa, ki se jo da generirati z največ štirimi elementi in poseduje abelovo podgrupo edinko indeksa največ p^4.*

Gornje restrikcije uporabimo za klasifikacijo B_0-minimalnih grup razreda nilpotentnosti 2.

Izrek (glej Theorem 4.45)**.** *Naj bo G B_0-minimalna grupa razreda nilpotentnosti* 2*. Tedaj je G izkolina eni od naslednjih grup:*

$$G_1 = \left\langle \; a, b, c, d \; \middle| \; \begin{array}{l} a^p = b^p = c^p = d^p = 1, \\ [a,b] = [c,d],\ [b,d] = [a,b]^{\varepsilon}[a,c]^{\omega},\ [a,d] = 1,\ \textit{class } 2 \end{array} \right\rangle,$$

$$G_2 = \left\langle \; a, b, c, d \; \middle| \; \begin{array}{l} a^p = b^p = c^p = d^p = 1, \\ [a,b] = [c,d],\ [a,c] = [a,d] = 1,\ \textit{class } 2 \end{array} \right\rangle,$$

kjer je $\varepsilon = 1$ za $p = 2$ in $\varepsilon = 0$ za liha praštevila p, število ω pa je generator grupe $(\mathbb{Z}/p\mathbb{Z})^{\times}$. Grupi G_1 in G_2 sta moči p^7 in njuna multiplikatorja Bogomolova sta $\mathrm{B}_0(G_1) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, $\mathrm{B}_0(G_2) \cong \mathbb{Z}/p\mathbb{Z}$.

Meje verjetnosti komutiranja

Struktura multiplikatorja Bogomolova je močno odvisna od komutirajočih parov dane grupe. Problem trivialnosti tega objekta zato lahko študiramo iz asimptotskega vidika glede na delež komutirajočih parov. V razdelku 5.1 raziskujemo verjetnost, da naključno izbrani par elementov končne grupe G komutira; to je število

$$\mathrm{vk}(G) = \frac{|\{(x,y) \in G \times G \mid [x,y] = 1\}|}{|G|^2}.$$

Raziščemo nekaj osnovnih lastnosti verjetnosti komutiranja. Najpomembnejša je zveza s številom razredov za konjugiranje $\mathrm{k}(G)$ grupe G.

Izrek (glej Theorem 5.2)**.** *Naj bo G končna grupa. Tedaj je* $\mathrm{vk}(G) = \mathrm{k}(G)/|G|$.

Število $\mathrm{vk}(G)$ je merilec komutativnosti grupe G. V tem smislu lahko na omejenost verjetnosti komutiranja $\mathrm{vk}(G)$ stran od 0 gledamo kot na lastnost, ki zagotavlja, da je grupa G bolj ali manj blizu abelovi grupi in zatorej približno poseduje lastnosti take grupe. To idejo najprej raziščemo v Razdelku 5.2, kjer poiščemo eksplicitno spodnjo mejo za verjetnost komutiranja, ki zagotovi trivialnost multiplikatorja Bogomolova. To najprej napravimo za p-grupe.

Izrek (glej Theorem 5.7)**.** *Naj bo G končna p-grupa. Če je*

$$\mathrm{vk}(G) > \frac{2p^2 + p - 2}{p^5},$$

potem je $\mathrm{B}_0(G)$ trivialen.

Z lokalnimi strukturnimi rezultati nato dobimo mejo za vse končne grupe.

Izrek (see Corollary 5.8)**.** *Naj bo G končna grupa. Če je $\mathrm{vk}(G) > 1/4$, potem je $\mathrm{B}_0(G)$ trivialen.*

Dokaz je precej zapleten. Študiramo minimalne protiprimere, ki so zaradi strukturnih lastnosti verjetnosti komutiranja in multiplikatorja Bogomolova nujno B_0-minimalne grupe. Uporabimo restriktivne rezultate o zgradbi teh grup. Nadaljevanje dokaza nato sloni na omejevanju moči centralizatorjev generatorjev take grupe. Splošno situacijo s podrobnim študijem komutatorske strukture zreduciramo na nekaj izoklinih družin s predstavniki majhnih moči. Te obdelamo z znanimi rezultati in aplikacijo razvitega računalniškega algoritma.

Z uporabo absolutne meje verjetnosti komutiranja izpeljemo neverjetnosten kriterij za trivialnost multiplikatorja Bogomolova.

Izrek (glej Corollary 5.10)**.** *Naj bo G končna grupa. Če je $|[G,G]|$ brez kubov, potem je $\mathrm{B}_0(G)$ trivialen.*

Zgornjo mejo verjetnosti komutiranja uporabimo tudi za konstrukcijo primerov B_0-minimalnih grup poljubno velikega razreda nilpotentnosti. Ti primeri nasprotujejo delom [Bog87, Theorem 4.6, Lemma 5.4].

V razdelku 5.3 najprej poenotimo verjetnost komutiranja in OK-razširitve z naslednjo trditvijo.

Izrek (glej Propostion 5.12)**.** *Centralna razširitev*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

je OK-razširitev natanko tedaj, ko velja $\mathrm{vk}(G) = \mathrm{vk}(Q)$.

Z uporabo teorije OK-krovov nato pridelamo meje velikosti multiplikatorja Bogomolova v odvisnosti od strukture dane grupe.

Izrek (glej Proposition 5.16)**.** *Naj bo Q končna grupa in S njena podgrupa edinka z lastnostjo, da je grupa Q/S ciklična. Tedaj* $|\mathrm{B}_0(Q)|$ *deli* $|\mathrm{B}_0(S)| \cdot |S^{\mathrm{ab}}|$ *in velja* $\mathrm{d}(\mathrm{B}_0(Q)) \leq \mathrm{d}(\mathrm{B}_0(S)) + \mathrm{d}(S^{\mathrm{ab}})$.

Izrek (glej Proposition 5.17)**.** *Naj bo Q končna grupa in S njena podgrupa. Tedaj se grupa* $\mathrm{B}_0(Q)^{|Q:S|}$ *vloži v* $\mathrm{B}_0(S)$.

Gornje strukturne meje omogočijo, da izpeljemo relativno različico rezultata o absolutni meji verjetnosti komutiranja. Tako omejimo rang in eksponent multiplikatorja Bogomolova pogojno na omejitev verjetnosti komutiranja.

Izrek (glej Theorem 5.20)**.** *Naj bo $\epsilon > 0$ in Q grupa z* $\mathrm{vk}(Q) > \epsilon$. *Tedaj lahko* $|\mathrm{B}_0(Q)|$ *omejimo z neko funkcijo, odvisno le od ϵ and* $\max\{\mathrm{d}(S) \mid S \text{ podgrupa Sylowa grupe } Q\}$. *Še več,* $\exp \mathrm{B}_0(Q)$ *lahko omejimo s funkcijo, odvisno le od ϵ.*

Nazadnje izpostavimo še zanimivo posledico v zvezi z eksponentom Schurovega multiplikatorja.

Izrek (glej Corollary 5.21)**.** *Ob danem $\epsilon > 0$ obstaja taka konstanta $C = C(\epsilon)$, da za vsako grupo Q z* $\mathrm{vk}(Q) > \epsilon$ *velja* $\exp M(Q) \leq C \cdot \exp Q$.

Vnovič racionalnost

V tem zadnjem poglavju raziščemo in uporabimo še eno znano pojavitev multiplikatorja Bogomolova. Tako podamo negativen odgovor na Isaacsovo domnevo o stopnjah karakterjev nekih grup. V razdelku 6.1 pričnemo tako, da najprej razširimo kontekst in opazujemo unitarne upodobitve Liejevih grup. *Kirillova metoda orbit* je znana strategija, ki v določenih primerih omogoča parametrizacijo upodobitev dane grupe z orbitami nekega delovanja. Ideja metode sloni na konceptu kvantizacije iz matematične fizike.

Metodo podrobneje predstavimo za razred *grup iz algeber*. To so grupe oblike $1+A$, kjer je A nilpotentna asociativna algebra nad poljem $\mathbb{F}$, tipično končnim. Takšna grupa je kot množica enaka $\{1+a \mid a \in A\}$, v njej pa množico z naravnim predpisom $(1+a)(1+b) = 1+a+b+ab$. Prototip grupe iz algebre je grupa $\mathrm{UT}_n(q)$. Teorija upodobitev teh grup je bogata, težka in zanimiva. Razumevanje sloni na naravni korespondenci med elementi grupe $1+A$ in njene Liejeve algebre A. Pod predpostavko $A^p = 0$ sta preslikavi

$$\exp\colon A \to 1+A,\ a \mapsto \sum_{i=0}^{\infty} \frac{a^i}{i!}, \quad \log\colon 1+A \to A,\ 1+a \mapsto \sum_{i=1}^{\infty} (-1)^{i+1} \frac{a^i}{i}$$

dobro definirani in ohranjata naravno delovanje grupe $1+A$ s konjugiranjem. V tem primeru Kirillova metoda orbit deluje in teorijo upodobitev takšne grupe iz algebre razumemo.

Brez predpostavke $A^p = 0$ eksponentna in logaritemska funkcija nista dobro definirani. Tako ni jasno, do kakšne mere Kirillova metoda orbit odpove za splošne grupe iz algeber. Isaacs je domneval, da bi metoda morala delovati vsaj na nivoju stopenj karakterjev. Stopnje kvazi-upodobitev, ki izhajajo iz orbit delovanja dane grupe, se imenujejo *lažne stopnje.* Domneva o lažnih stopnjah trdi, da so lažne stopnje enakem pravim. Domnevo reduciramo na naslednje vprašanje.

Vprašanje (glej Question 6.6)**.** *Naj bo A asociativna nilpotentna algebra nad končnim poljem. Ali je res, da je velikost abelacije grupe $1+A$ enaka velikosti abelacije Liejeve algebre A?*

Odgovora na vprašanje se lotimo v Razdelku 6.2 s študiranjem grup, ki izhajajo iz modularnih grupnih kolobarjev $\mathbb{F}_q[X]$ končnih p-grup X. Za takšne grupe iz algeber je lahko izračunati Liejevo abelacijo ustrezne algebre, ki je v tem primeru enaka augmentacijskemu idealu $\mathrm{I}_{\mathbb{F}_q}$.

Izrek (glej Lemma 6.7)**.** *Naj bo X končna grupa in $\mathbb{F}$ polje. Tedaj je*

$$\dim_{\mathbb{F}} \mathrm{I}_{\mathbb{F}} / [\mathrm{I}_{\mathbb{F}}, \mathrm{I}_{\mathbb{F}}]_L = \mathrm{k}(X) - 1.$$

Bistveno težje pa je izračunati velikost abelacije grupe obrnljivih elementov $1+\mathrm{I}_{\mathbb{F}_q}$. Problem najprej dvignemo v ničelno karakteristiko tako, da polje $\mathbb{F}_q$ menjamo z ustrezno razširitvijo kolobara p-adičnih celih števil $\mathrm{R}_q = \mathbb{Z}_p[\zeta_{q-1}]$, kjer je ζ_{q-1} primitiven $(q-1)$-ti koren enote. V tem novem okolju lahko problem izrazimo v jeziku K-teorije, saj za lokalen kolobar R velja $\mathrm{K}_1(\mathrm{R}) = \mathrm{R}^*_{\mathrm{ab}}$. Naš pristop tako sloni na povezavi med $\mathrm{K}_1(\mathbb{F}_q[X])$ in $\mathrm{K}_1(\mathrm{R}_q[X])$. V ničelni karakteristiki so prve K-teoretične grupe dobro raziskane. Podrobneje, naj bo Q_q kolobar kvocientov R_q. Postavimo

$$\mathrm{SK}_1(\mathrm{R}_q[X]) = \ker\Big(\mathrm{K}_1(\mathrm{R}_q[X]) \to \mathrm{K}_1(\mathrm{Q}_q[X])\Big).$$

Problem razumevanja obstoja eksponentne in logaritemske preslikave se v ničelni karakteristiki izraža kot problem trivialnosti objekta $\mathrm{SK}_1(\mathrm{R}_q[X])$. Slednjega je nad p-adičnimi kolobarji raziskal Oliver [Oli80] in v našem jeziku dokazal naslednje.

Izrek (glej Theorem 6.9)**.** *Obstaja naraven izomorfizem* $\mathrm{SK}_1(\mathrm{R}_q[X]) \cong \mathrm{B}_0(X)$.

Multiplikator Bogomolova torej predstavlja obstrukcijo k problemu obstoja dobre korespondence med grupo iz algebre in njeno Liejevo algebro. S projiciranjem tega rezultata prek K-teoretičnih sredstev uspemo izračunati velikost grupne abelacije $(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$.

Izrek (glej Theorem 6.10)**.** *Naj bo X končna p-grupa. Tedaj je*

$$|(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}| = q^{\mathrm{k}(X)-1}|\mathrm{B}_0(X)|.$$

Dokaz temelji na strukturni dekompoziciji grupe $(1 + \mathrm{I}_{\mathbb{F}_q})_{\mathrm{ab}}$, od koder gornja trditev o močeh neposredno sledi. Ko vzamemo za X grupo z netrivialnim multiplikatorjem Bogomolova, na ta način ovržemo Isaacsovo domnevo o lažnih stopnjah v vseh karakteristikah.

Zaključimo z Razdelkom 6.3, kjer najdemo konceptualno razlago za neregularno obnašanje nad končnimi polji. Namesto dviga prvotnega problema v ničelno karakteristiko ga tu obravnavamo nad algebraičnim zaprtjem $\mathbb{F}$ polja $\mathbb{F}_p$. Na grupo $\mathbf{G} = 1 + \mathrm{I}_{\mathbb{F}}$ lahko gledamo kot na algebraično grupo, definirano nad $\mathbb{F}_p$. Pišimo $\mathbf{G}(\mathbb{F}_q) = 1 + \mathrm{I}_{\mathbb{F}_q}$ za grupo $\mathbb{F}_q$-točk algebraične grupe $\mathbf{G}$. Če želimo opazovati izpeljano podgrupo, je potrebno biti previden, saj se v splošnem grupi $\mathbf{G}(\mathbb{F}_q)'$ in $\mathbf{G}'(\mathbb{F}_q)$ razlikujeta. Jasno drži inkluzija

$$(1 + \mathrm{I}_{\mathbb{F}_q})' = (\mathbf{G}(\mathbb{F}_q))' \subseteq \mathbf{G}'(\mathbb{F}_q),$$

ki pa ni nujno enakost. Razliko na tem končnem nivoju meri multiplikator Bogomolova.

Izrek (glej Theorem 6.14)**.** *Naj bo X končna p-grupa in* $\mathbf{G} = 1 + \mathrm{I}_{\mathbb{F}}$.

1. *Velja*
$$|\mathbf{G}(\mathbb{F}_q) : \mathbf{G}'(\mathbb{F}_q)| = q^{\mathrm{k}(X)-1}.$$

2. *Velja*
$$\mathbf{G}'(\mathbb{F}_q)/\mathbf{G}(\mathbb{F}_q)' \cong \mathrm{B}_0(X).$$

Dokaz temelji na primerjanju grupnih abelacij racionalnih točk $\mathbf{G}(\mathbb{F}_q)_{\mathrm{ab}}$ za različne razširitve polja $\mathbb{F}_q$. Te zopet interpretiramo kot prve K-teoretične grupe in se skličemo na strukturno dekompozicijo, izpeljano poprej v končnem primeru. Tako vstopi multiplikator Bogomolova. Da določimo dimenzijo $\dim \mathbf{G}'$, upoštevamo še Liejevo algebro grupe

$\mathbf{G}$. Nazadnje dejstvo, da multiplikator Bogomolova obstane v konstrukciji, preberemo s pomočjo K-teoretičnega prenosa.

Druga izjava gornje trditve ponuja novo interpretacijo multiplikatorja Bogomolova. Tako lahko gledamo na ta objekt kot na komutatorje, ki se v $\mathbf{G}$ evalvirajo v racionalno točko, niso pa sami sestavljeni iz racionalnih točk. To nas spominja na samo vpeljavo multiplikatorja Bogomolova ter na motivacijski izvor. Z uporabo nove interpretacije pridobimo svež pogled na eksponent multiplikatorja Bogomolova.

Izrek (glej Theorem 6.15)**.** *Naj bo X končna p-grupa in $\mathbf{G} = 1 + \mathrm{I}_{\mathbb{F}}$. Velja*

$$\exp \mathrm{B}_0(X) = \min\{m \mid \mathbf{G}'(\mathbb{F}_q) \subseteq \mathbf{G}(\mathbb{F}_{q^m})'\}.$$